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MATHEMATICS 31

MODULE 2

Limits

Distance Learning

Alberta EDUCATION

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Mathematics 31

Module 2

LIMITS

Distance Learning

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Teachers (Mathematics 31)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 31
Student Module Booklet
Module 2
Limits
Alberta Distance Learning Centre
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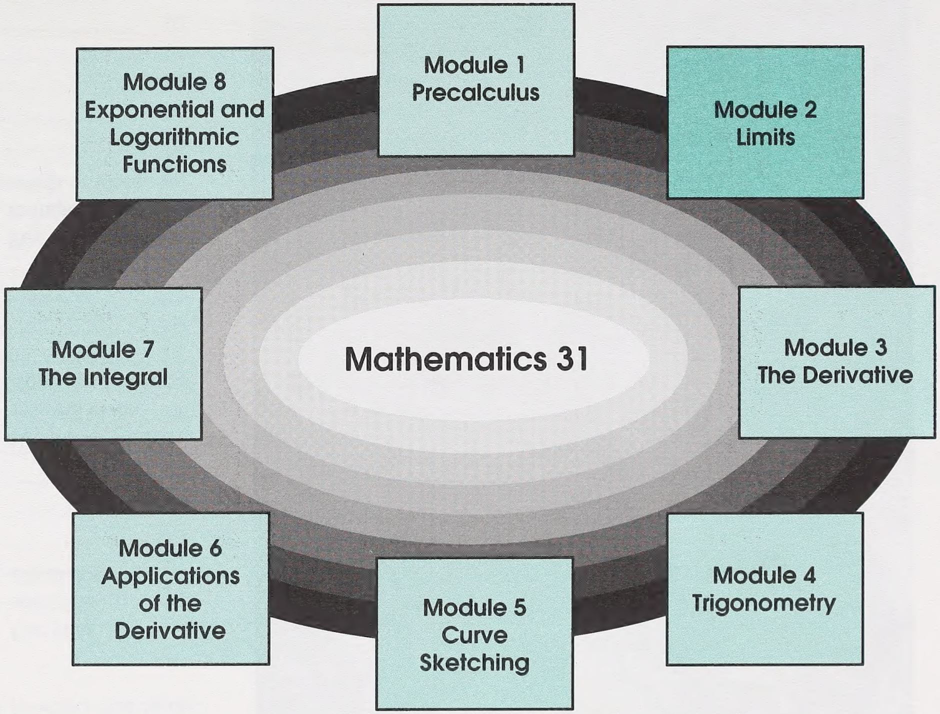
Welcome

WESTFILE INC.

Welcome to Module 2. We hope you'll enjoy your study of Limits.

Mathematics 31 contains eight modules. Work through modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

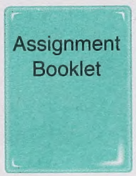
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

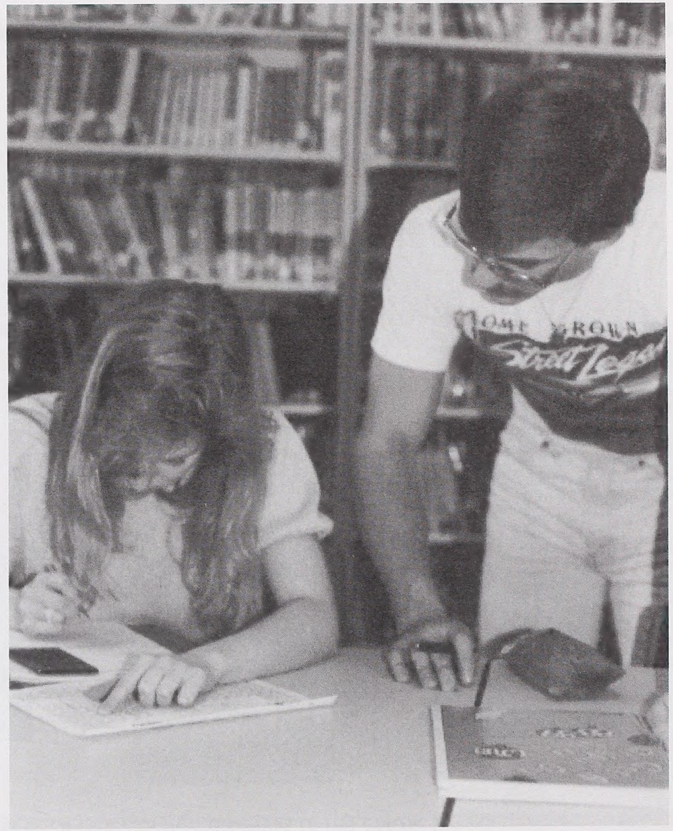
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the assignment booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



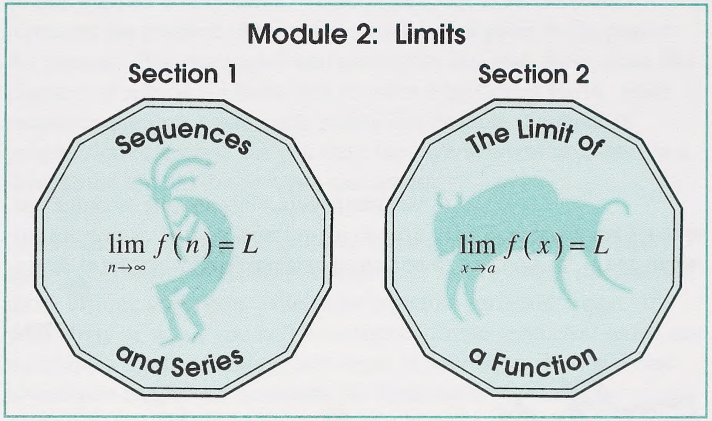
There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

CONTENTS

Module Overview

Often a coin appears quite different when it is flipped over and examined from the opposite side. Of course, turning the coin over does not change the coin, it just changes your perspective. When you see a dollar coin, it does not matter which side you view, you know immediately what its value is!

The notion of limit is fundamental to your understanding of either differential or integral calculus. This module introduces you to the topic of limits from two complementary points of view, like two sides of the same coin.



In Section 1, sequences and series are used to develop the concept of limit and to lay a basis for finding limits numerically and geometrically. In particular, Section 1 deals with convergent infinite sequences and their limits. Section 1 also extends the concept of series introduced in Mathematics 30 to include infinite geometric series, techniques for evaluating those series, and applying them to everyday situations.

Section 2 uses algebraic functions and their graphs to develop and extend your understanding of limits, thereby enhancing your ability to analyse functions and their graphs. Some of the skills you will develop in Section 2 are defining limits, determining left- and right-hand limits and whether the graphs of particular functions are continuous or discontinuous, using limit theorems to find limits of algebraic functions, and inspecting curves at their extremes.

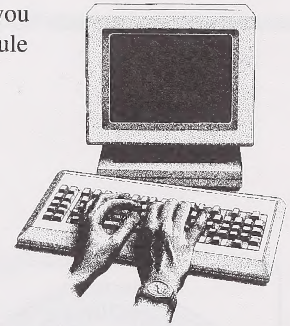
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete two section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**32 marks**
Section 2 Assignment	**46 marks**
Final Module Assignment	**22 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

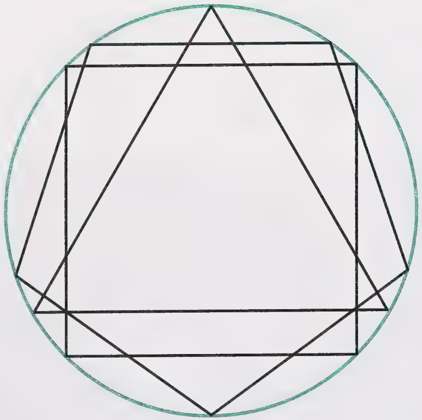
There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Sequences and Series

Have you ever visited the Badlands in Alberta? If you have, you may have noticed that rivers and streams have cut into that landscape over time revealing layer upon layer of deposits and the fossils trapped within them. These layers, one after the other, represent the passage of time, from millions of years in the past to the present. Paleontologists and geologists can read those strata like chapters of a book—a book that contains a history of Earth. Each stratum represents a particular period in a definite sequence or progression of events. As you trace the layers downward, there is a limit to the time in the past you can access.

In this section you will be studying sequences and series and their limits. You will sketch infinite sequences and calculate the limits of those infinite sequences which converge to a particular value. In your study of series you will investigate infinite geometric series and techniques for determining their sums, if they exist. You will also investigate conditions necessary for finite sums of infinite geometric series. You will be applying what you learn to geometric figures, to changing repeating decimal numerals into fractions, and to everyday events such as determining how far a ball will bounce before coming to rest.

Activity 1: Limit of a Sequence



If you were to calculate the area of each polygon inscribed in the circle, you would notice that the area increases as the number of sides increases. The area of the triangle is less than the area of the square, which, in turn, is less than the area of the pentagon, and so on.

These areas form an increasing **sequence**. But, is there any **limit** to the areas of these polygons? Could the area of any of these polygons ever exceed the area of the circle itself?

Infinite sequences and their limits are basic to the understanding of calculus. Even though sequences are more fully developed in Mathematics 30, there are a number of key concepts you will need to know for Mathematics 31.

An **infinite sequence** is the range of a function which has the set of natural numbers as its domain.

Example 1

Determine the first five terms of the sequence defined by the function $f(n) = 2n - 1,\ n \in N$.

Solution

n	1	2	3	4	5
$f(n)$	1	3	5	7	9

The preceding table of values illustrates that the domain of this function is the set of natural numbers *N*. The values of $f(n)$ are 1, 3, 5, 7, and 9 (the first five terms in the sequence). Particular terms of the sequence can be represented using functional notation or subscripts.

First term: $f(1) = t_1$
$= 1$

Second term: $f(2) = t_2$
$= 3$

Third term: $f(3) = t_3$
$= 5$

*n*th term or general term: $f(n) = t_n$
$= 2n - 1$

Example 2

Determine a possible general term for the sequence 2, 5, 10, 17, 26, ….

Solution

The terms in the sequence may be represented as 1^2+1, 2^2+1, 3^2+1, 4^2+1, and 5^2+1. The function generating those terms is $f(n)=n^2+1,\ n\in N$, or $t_n=n^2+1,\ n\in N$.

1. Give the first five terms for each sequence given.

   a. $f(n)=5n-3,\ n\in N$

   b. $f(n)=2n^2+1,\ n\in N$

   c. $t_n=3+\frac{2}{n},\ n\in N$

   d. $t_n=3^n+1,\ n\in N$

2. Find the stated term of the given sequence.

   a. third term of $f(n)=-n^3,\ n\in N$

   b. eighth term of $t_n=2^{-n}-1,\ n\in N$

   c. ninth term of $f(n)=3-8n,\ n\in N$

Check your answers by turning to the Appendix.

Example 3

Graph the terms obtained from the following sequences:

- $f(n)=\frac{n^2}{3},\ n\in N$
- $f(n)=\frac{n-1}{n},\ n\in N$
- $t_n=(-3)^{-n},\ n\in N$

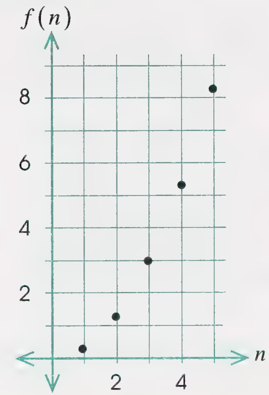
Solution

For $f(n)=\frac{n^2}{3}$, the terms of the sequence are given in the following table of values.

n	1	2	3	4	5
$f(n)$	$\frac{1}{3}$	$\frac{4}{3}$	3	$\frac{16}{3}$	$\frac{25}{3}$

Here is a sample calculation:

$$f(1)=\frac{1^2}{3}$$
$$=\frac{1}{3}$$

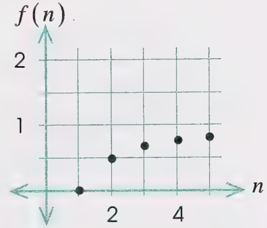


For $f(n)=\frac{n-1}{n}$, the terms of the sequence are given in the following table of values.

n	1	2	3	4	5
$f(n)$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$

Here is a sample calculation:

$$\begin{aligned} f(1) &= \frac{1-1}{1} \\ &= 0 \end{aligned}$$

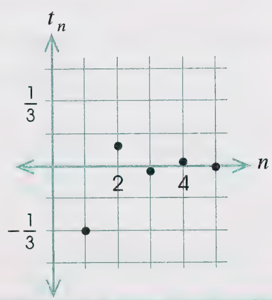


For $t_n = (-3)^{-n}$, the terms of the sequence are given in the following table of values.

n	1	2	3	4	5
t_n	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{1}{27}$	$\frac{1}{81}$	$-\frac{1}{243}$

Here is a sample calculation:

$$\begin{aligned} t_1 &= (-3)^{-1} \\ &= -\frac{1}{3} \end{aligned}$$



Remember that each function is defined for natural numbers only. Do not join the points on the graph; that would suggest that n is replaceable by any real value.

For the sequence $f(n)=\frac{n^2}{3}$, $n \in N$, the values of the terms $f(n)$ increase without bound as n (position of the term in the sequence) increases without bound. As $n \to \infty$, $f(n) \to \infty$.

For instance, when $n = 1000$, $f(n) \doteq 333\,333.3$.

However, for $f(n)=\frac{n-1}{n}$, $n \in N$, and $t_n = (-3)^{-n}$, $n \in N$, the terms approach a unique finite value, called the **limit**, as n increases.

As seen from their graphs, the limit for $f(n)=\frac{n-1}{n}$, $n \in N$, is 1; and the limit for $t_n = (-3)^{-n}$, $n \in N$, is 0.

Symbolically, these limits are written as follows:

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = 1 \text{ and } \lim_{n\to\infty}(-3)^{-n} = 0$$

If the terms of an infinite sequence approach a unique finite value, that sequence is called a **convergent sequence**. A sequence which does not converge is called **divergent**.

Example 4

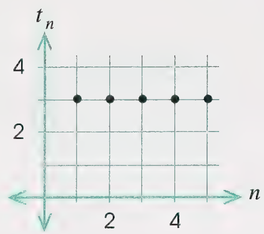
Graph each of the following sequences. Which sequences converge? State the limit for each convergent sequence.

- $t_n = 3,\ n \in N$
- $f(n) = \dfrac{2n}{n+1},\ n \in N$
- $t_n = (-1)^n \quad, n \in N$

Solution

$t_n = 3$ is a convergent sequence.

n	1	2	3	4	5
t_n	3	3	3	3	3



$$\lim_{n\to\infty} 3 = 3$$

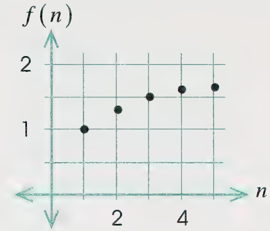
Here, the terms of the sequence equal the limit.

$f(n) = \frac{2n}{n+1}$ is a convergent sequence.

n	1	2	3	4	5
$f(n)$	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{5}{3}$

Here is a sample calculation:

$$\begin{aligned} f(1) &= \frac{2(1)}{1+1} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$

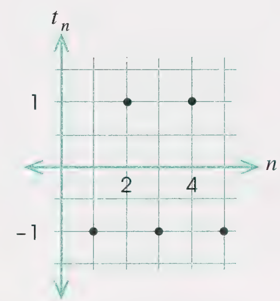


$$\lim_{n\to\infty}\left(\frac{2n}{n+1}\right) = 2$$

From the graph, the terms appear to approach 2. Use a calculator to evaluate $f(9)$, $f(99)$, $f(999)$, $f(9999)$. Did you obtain 1.8, 1.98. 1.998, and 1.9998 respectively? Even though the terms in the sequence never equal 2, they can be made arbitrarily close to 2 simply by choosing sufficiently large values of *n*.

$t_n = (-1)^n$ is a divergent sequence.

n	1	2	3	4	5
t_n	−1	1	−1	1	−1



Since the terms alternate between 1 and −1 regardless of the size of *n*, the limit is undefined.

Use a calculator to answer question 3.

3. Do the following sequences appear to converge or diverge? State what you feel is the limit for each convergent sequence. Test your conjectures by selecting larger and larger values of n to evaluate the corresponding terms of the sequence.

a. $f(n)=\frac{6-n}{n-4}$, where $n \in N$

b. $f(n)=n-2$, where $n \in N$

c. $f(n)=\frac{1}{n}$, where $n \in N$

Check your answers by turning to the Appendix.

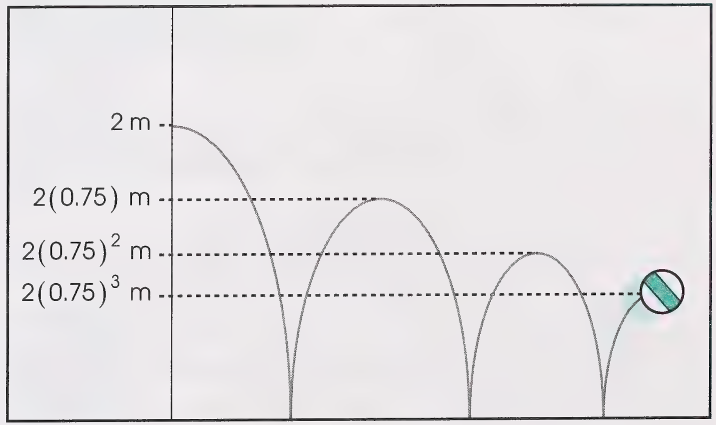
If you generalize the results of question 3, you will observe that $\lim_{n\to\infty} \frac{1}{n}=0$ and $\lim_{n\to\infty} r^n=0$, where $|r|<1$.

These results provide the basis for an algebraic approach for determining limits of inifinite sequences. You will use these results in Example 5.

Example 5

A ball is dropped from a height of 2 m above flat ground. Each time it hits the ground, after falling a distance h, it rebounds $0.75h$. The heights to which the ball rises form a sequence. What is that sequence, and what is its limit?

Solution



On the first bounce, the ball rises $(0.75)(2)=1.5$ m.

On the second bounce, the ball rises $(0.75)(1.5)=(2)(0.75)^2=1.125$ m.

On the third bounce, the ball rises $(0.75)(1.125)=(2)(0.75)^3=0.843\,75$ m.

These terms form a geometric sequence.

$$t_n = 2(0.75)^n, \text{ where } n \in N$$

As the ball continues to bounce, the height to which it rises approaches zero, the limit of the sequence.

As noted earlier, $\lim\limits_{n\to\infty} r^n = 0$ if $|r|<1$. In this example, $r = 0.75$.

This is more evident if you evaluate $(0.75)^n$ for $n = 10$.

$$(0.75)^{10} \doteq 0.056 \text{ m}$$

Algebraically, $\lim\limits_{n\to\infty}(0.75)^n = 0$ and $\lim\limits_{n\to\infty}(2)(0.75)^n = 0$.

Example 6

Determine the limit of the sequence $f(n) = \frac{2n-3}{n}$ as $n \to \infty$.

Solution

$$\begin{aligned}\lim_{n\to\infty}\frac{2n-3}{n} &= \lim_{n\to\infty}\left(\frac{2n}{n}-\frac{3}{n}\right)\\ &= \lim_{n\to\infty}\left(2-\frac{3}{n}\right)\\ &= 2-0\\ &= 2\end{aligned}$$

Separate into fractions.

As $n \to \infty$, $\frac{3}{n} \to 0$.

Example 7

Determine the limit of the following sequence as $n \to \infty$.

$$f(n) = \frac{3n^2-5n+8}{2n^2+3n-7}$$

Solution

$$\begin{aligned}\lim_{n\to\infty}\frac{3n^2-5n+8}{2n^2+3n-7} &= \lim_{n\to\infty}\frac{\cancel{n^2}\left(3-\frac{5}{n}+\frac{8}{n^2}\right)}{\cancel{n^2}\left(2+\frac{3}{n}-\frac{7}{n^2}\right)}\\ &= \lim_{x\to\infty}\frac{3-\frac{5}{n}+\frac{8}{n^2}}{2+\frac{3}{n}-\frac{7}{n^2}}\\ &= \frac{3}{2}\end{aligned}$$

Remove n^2 as a factor.

As $n \to \infty$, $\frac{5}{n}$, $\frac{8}{n^2}$, $\frac{3}{n}$, and $\frac{7}{n^2}$ all approach 0.

In Example 7, the highest power of n was removed as a common factor in order to obtain fractions such as $\frac{5}{n}$, $\frac{8}{n^2}$, and so on. These fractions tend to 0 as n increases without bound. Remember, not all sequences have finite limits.

Example 8

Calculate $\lim\limits_{n\to\infty} \dfrac{n^3+1}{n^2-2}$.

Solution

$$\lim_{n\to\infty} \frac{n^3+1}{n^2-2} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n^3}\right)}{\cancel{n^3}\left(\frac{1}{n}-\frac{2}{n^3}\right)}$$

$$= \frac{1+0}{0-0}$$

$= \text{undefined}$

n^3 is the highest power of n in the limit.

Notice that as n increases without bound, $\frac{n^3+1}{n^2-2}$ also increases without bound. Here, you may write $\lim\limits_{n\to\infty} \frac{n^3+1}{n^2-2} = \infty$.

4. Find the limit of each of the following sequences as $n \to \infty$.

a. $\lim\limits_{n\to\infty} \dfrac{n+3}{n}$

b. $\lim\limits_{n\to\infty} \dfrac{5+3n}{n}$

c. $\lim\limits_{n\to\infty} \dfrac{3n^3-2n+1}{4n^3-2n^2+n-7}$

d. $\lim\limits_{n\to\infty} \dfrac{n^3-8n^2+3n+1}{n^3-2n+8}$

e. $\lim\limits_{n\to\infty} \dfrac{n^2+2}{n^3-1}$

f. $\lim\limits_{n\to\infty} \dfrac{n^3}{n^2+2}$

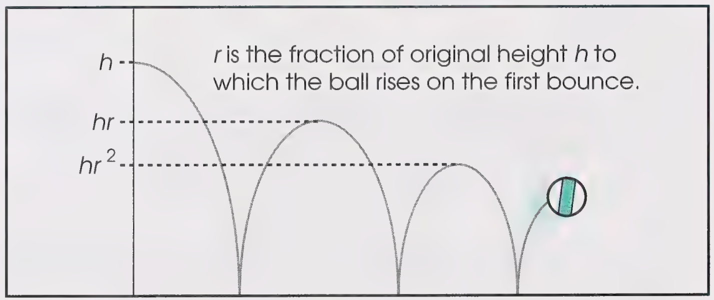
5. If the degree of the numerator is less than the degree of the denominator, what can you conclude about $\lim\limits_{n\to\infty} f(n)$? What can you conclude if the degree of the denominator is less than the degree of the numerator?

Check your answers by turning to the Appendix.

You should now be able to find the limit of a sequence. At the beginning of this activity you looked at polygons inscribed in a circle. Do you agree that for a circle of radius r,

$\lim\limits_{n\to\infty}(\text{area of inscribed } n\text{-gon}) = \pi r^2$?

Activity 2: Limit of a Series



Recall from Activity 1 that the terms representing the heights to which a ball bounces form the terms of a geometric sequence. How would you determine the total distance that the ball would bounce before it comes to rest?

Example 1

A ball is dropped from a height of 2 m. On each bounce it rises three-quarters of the height from which it falls. Assuming it continues in this fashion until it comes to rest, what is the total distance the ball travels?

Solution

The distance the ball travels during each bounce forms a geometric sequence.

$$\text{First bounce} = 0.75(2)\ \text{m up} + 0.75(2)\ \text{m down}$$
$$= 3\ \text{m}$$

$$\text{Second bounce} = 3(0.75)\ \text{m}$$

$$\text{Third bounce} = 3(0.75)^2\ \text{m}$$

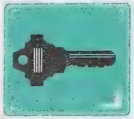
Don't forget that the ball was dropped from a height of 2 m. The task is to find the sum of all the terms or

$2+\left[3+3(0.75)+3(0.75)^2+\ldots\right].$

One approach is to use a calculator to find successive sums (adding distances one bounce at a time) and seeing if these partial sums approach a limit.

Number of Bounces	Total Distance Travelled (m)
0	2
1	5
2	7.25
3	8.9375
4	10.203 125
5	11.152 343 75
6	11.864 257 81
7	12.398 193 36
8	12.798 645 02
9	13.098 983 76
⋮	⋮
20	13.949 260 61
⋮	⋮
40	13.999 839 09
41	13.999 879 32
42	13.999 909 49

It appears that the partial sums approach a limit of 14 m, the total distance the ball travels before it comes to rest. Are you sure?



Recall from Mathematics 30, that a series is simply the sum of the terms of a sequence. An infinite series is the sum of the terms of an infinite sequence. In this section, you will be working primarily with infinite geometric series.

Infinite Geometric Sequence

$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots,\ ar^{n-1}$, where a is the first term and r is the common ratio

Infinite Geometric Series

$$S = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots$$

The sum of the first n terms is $S_n = \dfrac{a\left(1-r^n\right)}{1-r}$.

1. For each geometric sequence, state the first term a, and the common ratio r. Form an infinite geometric series from each sequence.

a. $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ldots$ **b.** $3,\ -6,\ 12,\ -24,\ 48$

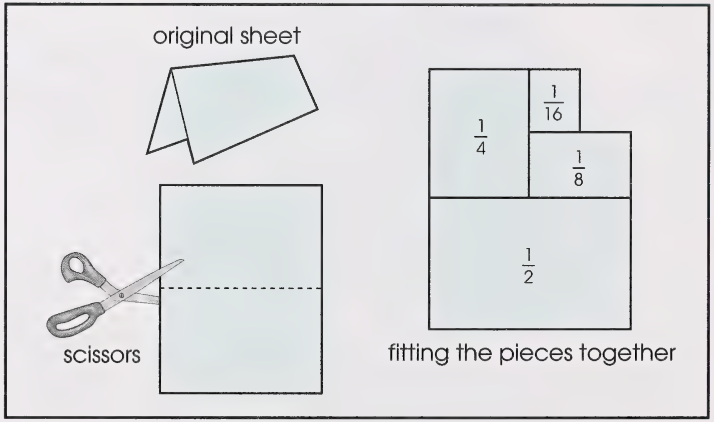
c. $3,\ -1,\ \frac{1}{3},\ -\frac{1}{9},\ldots$

Check your answers by turning to the Appendix.

You can use the formula $S_n = \frac{a\left(1-r^n\right)}{1-r}$ to find the sum of the first 10 terms, or 100 terms, or 1000 terms, but how would you find the sum of an infinite number of terms? Is that sum ever finite?

Example 2

A sheet of paper is folded in two, and cut along the crease; one of the two pieces is labelled as $\frac{1}{2}$. The other piece is folded in two, and cut along the crease; one of the two pieces is labelled as $\frac{1}{4}$. The other piece is folded in two, and cut along the crease; one of the two pieces is labelled $\frac{1}{8}$. This process may be continued indefinitely. Your task is to put the sheet of paper back together, as shown.



Explain why this task may be modelled by an infinite geometric series, and form a sequence of partial sums. Explain why the sequence of partial sums approaches the sum of the infinte series. Use a graph to support your answer. Then, calculate the sum of the infinite series.

Solution

The piece of paper is reassembled by adding the fractional parts.
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ is an infinite geometric series with $a=\frac{1}{2}$ and $r=\frac{1}{2}$.

The following is a sequence of partial sums.

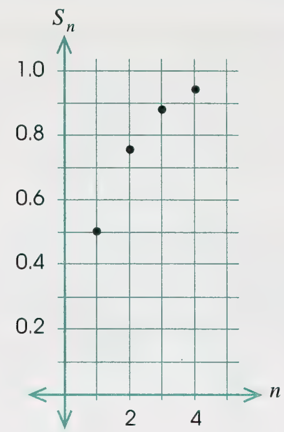
$$S_1 = \text{first term} \qquad\qquad S_2 = \text{sum of the first two terms}$$
$$= \frac{1}{2} \qquad\qquad = \frac{1}{2}+\frac{1}{4}$$
$$= \frac{3}{4}$$

$$S_3 = \text{sum of the first three terms} \qquad S_4 = \text{sum of the first four terms}$$
$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{8} \qquad = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$$
$$= \frac{7}{8} \qquad = \frac{15}{16}$$

The sequence of partial sums S_1, S_2, S_3, and S_4 is $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{15}{16}$.

As more and more pieces are added in the figure, the partial sum approaches the sum of the infinite series.

$$S_n \to S \text{ as } n \to \infty$$



From the graph of the sequence of partial sums, the actual sum (or limiting value) must be 1 (the size of the original piece of paper). The sum of the infinite series is as follows:

You must show that $\lim_{n\to\infty} S_n = 1$.

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \\
&= \lim_{n\to\infty} \left[1-\left(\frac{1}{2}\right)^n\right] \qquad \left(\text{since } \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^n = 0\right) \\
&= 1-0 \\
&= 1
\end{aligned}$$

The preceding example should clarify the development of the formula for the sum of the terms of an infinite geometric series for which the common ratio r satisfies the condition $|r|<1$. The development of the formula follows.

Given the infinite geometric series $S=a+ar+ar^2+ar^3+ar^4+\ldots+ar^{n-1}+\ldots$, the sum of the first n terms is $S_n=\frac{a\left(1-r^n\right)}{1-r}$ if $r\neq 1$ and

$S=\lim\limits_{n\to\infty} S_n=\lim\limits_{n\to\infty}\frac{a\left(1-r^n\right)}{1-r}$. Remember, if $|r|<1$, as $n\to\infty$, then $r^n\to 0$. Therefore, $S=\frac{a}{1-r}$.

2. Verify the results of Examples 1 and 2, using the formula $S=\frac{a}{1-r}$.

Check your answers by turning to the Appendix.

Example 3

Calculate $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\ldots$.

Solution

$a=2$ and $r=-\frac{1}{3}$

$$\begin{aligned}S&=\frac{a}{1-r}\\&=\frac{2}{1-\left(-\frac{1}{3}\right)}\\&=\frac{2}{\frac{4}{3}}\\&=\frac{3}{2}\end{aligned}$$

Example 4

Convert 0.121 212... and 1.324 5245... into fractions.

Solution

$0.121\,212\ldots=0.12+0.0012+0.000\,012+\ldots$

$\therefore\ a=0.12$ and $r=.01$

$$\begin{aligned}S&=\frac{a}{1-r}\\&=\frac{0.12}{1-0.01}\\&=\frac{12}{99}\\&=\frac{4}{33}\end{aligned}$$

$1.324\,5245\ldots = 1.3 + (0.0245 + 0.000\,0245 + \ldots)$

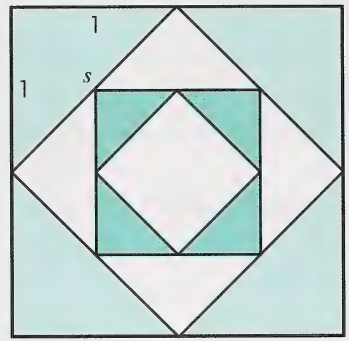
$\therefore\ a = 0.0245$ and $r = .001$

$$\begin{aligned} S &= 1.3 + \frac{a}{1-r} \\ &= 1.3 + \frac{0.0245}{1-0.001} \\ &= 1.3 + \frac{0.0245}{0.999} \\ &= 1.3 + \frac{245}{9990} \\ &= \frac{13\,232}{9990} \\ &= \frac{6616}{4995} \end{aligned}$$

Check the conversions in Example 4 using a calculator.

Example 5

In the following diagram, the midpoints of a square 2 units on a side are joined to form a second square. The midpoints of that square are joined to form a third square, and so on. What is the total perimeter of all such squares?



Solution

$$\begin{aligned} \text{Perimeter of first square} &= 4(2) \\ &= 8 \end{aligned}$$

Each side s of the second square can be found using Pythagoras's Theorem.

$$\begin{aligned} s^2 &= 1^2 + 1^2 \\ s^2 &= 2 \\ s &= \sqrt{2} \end{aligned}$$

Perimeter of second square $= 4\left(\sqrt{2}\right)$

$$= 4\sqrt{2}$$

Perimeter of third square $= 4$

Total Perimeter $P = 8 + 4\sqrt{2} + 4 + \ldots$

This series is geometric. The common ratio can be found by dividing the second term by the first term.

$$r = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2}$$

$$\therefore\ P = \frac{8}{1-\frac{\sqrt{2}}{2}} = \frac{8}{1-\frac{\sqrt{2}}{2}} \times \frac{2}{2} = \frac{16}{2-\sqrt{2}}$$

Multiply by $\frac{2}{2}$ to clear the fraction from the denominator.

Note: Answers should not be left with a radical denominator.

Since $(a-b)(a+b) = a^2 - b^2$, the radical in the denominator can be eliminated by multiplying top and bottom by $2+\sqrt{2}$; this expression is called the **conjugate expression** (or simply conjugate) of $2-\sqrt{2}$.

$$\therefore\ P = \frac{16}{\left(2-\sqrt{2}\right)} \times \frac{\left(2+\sqrt{2}\right)}{\left(2+\sqrt{2}\right)} = \frac{16\left(2+\sqrt{2}\right)}{2^2-\left(\sqrt{2}\right)^2} = \frac{16\left(2+\sqrt{2}\right)}{4-2} = 8\left(2+\sqrt{2}\right) = 16+8\sqrt{2}$$

3. Calculate the sum of each infinite geometric series, if it exists.

a. $4+2+1+\ldots$

b. $3-1+\frac{1}{3}-\frac{1}{9}+\ldots$

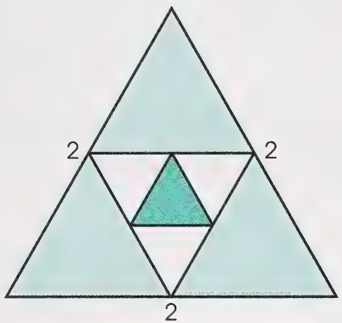
c. $6+4-2+1-\frac{1}{2}+\ldots$

d. $1+2+4+8+\ldots$

4. Convert each repeating decimal numeral to a fraction of the form $\frac{a}{b}$, where a and $b \in I$.

a. 0.121 212… **b.** 2.333… **c.** $6.123\,\overline{23}$

5. An equilateral triangle measures 2 units on a side. The midpoints of its sides are joined to form a second equilateral triangle. The midpoints of the sides of that triangle are joined, and so on. What is the total perimeter of all such triangles?



6. A ball at the end of a string is set to swing through a 30 cm arc. On its next swing, the arc through which it travels is 90% as large. If each swing is 90% of the previous swing, what distance does the ball travel before it comes to rest?

Check your answers by turning to the Appendix.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In Activity 2, the solution to Example 5 involved rationalizing a binomial denominator. If you are not familiar with this process, study the following examples.

Example 1

Rationalize $\dfrac{2}{\sqrt{3}-1}$.

Solution

Multiply both the numerator and the denominator by the conjugate of the denominator. The conjugate of $\sqrt{3}-1$ is $\sqrt{3}+1$, formed by changing the sign between the two terms in the original binomial.

Note: A conjugate is used because the product of the sum and difference of the same two terms is the difference of their squares, thus eliminating the root.

$$\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)=\left(\sqrt{3}\right)^2-1^2$$
$$=3-1$$
$$=2$$

$$\therefore \frac{2}{\sqrt{3}-1}=\frac{2\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$$
$$=\frac{2\left(\sqrt{3}+1\right)}{\left(\sqrt{3}\right)^2-1}$$
$$=\frac{2\left(\sqrt{3}+1\right)}{2}$$
$$=\sqrt{3}+1$$

The expression $\frac{\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)}$ is called a unit factor. Since this expression is equal to 1, it is used to change the form of the original expression, not its value.

Example 2

Rationalize $\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-2\sqrt{2}}$.

Solution

$$\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-2\sqrt{2}}=\frac{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}+2\sqrt{2}\right)}{\left(\sqrt{6}-2\sqrt{2}\right)\left(\sqrt{6}+2\sqrt{2}\right)}$$
$$=\frac{\sqrt{36}+2\sqrt{12}+\sqrt{12}+2\sqrt{4}}{\left(\sqrt{6}\right)^2-\left(2\sqrt{2}\right)^2}$$
$$=\frac{6+3\sqrt{12}+2(2)}{6-4(2)}$$
$$=\frac{10+3\sqrt{4(3)}}{-2}$$
$$=\frac{10+6\sqrt{3}}{-2}$$
$$=\frac{-2\left(-5-3\sqrt{3}\right)}{-2}$$
$$=-5-3\sqrt{3}$$

1. State the conjugate of each radical. Multiply the original binomial by its conjugate. What do you notice?

a. $\sqrt{3}+\sqrt{2}$ **b.** $2\sqrt{3}-\sqrt{7}$ **c.** $2\sqrt{5}+3\sqrt{7}$

2. Rationalize each of the following:

a. $\dfrac{6}{\sqrt{7}-2}$ b. $\dfrac{3+\sqrt{2}}{2+\sqrt{2}}$ c. $\dfrac{\sqrt{6}-2\sqrt{3}}{2\sqrt{6}+\sqrt{3}}$

Check your answers by turning to the Appendix.

Enrichment

From your work in this section you have found sums of infinite geometric series, which were whole numbers or fractions. For instance,

$$\begin{aligned}\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\ldots &= \frac{\frac{1}{3}}{1-\frac{1}{3}} \\ &= \frac{\frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2}\end{aligned}$$

Is it possible to reverse the process? That is, given a fraction or whole number, is it possible to express that value as an infinite geometric series?

Example

Express $\frac{2}{3}$ as an infinite series.

Solution

Since $\frac{2}{3}=0.666\ldots$, an obvious series is $\frac{2}{3}=0.6+0.06+0.006+\ldots$. However, the formula $S=\frac{a}{1-r}$ can be used to obtain more interesting results.

$$\begin{aligned}\frac{2}{3} &= \frac{0.2}{0.3} \quad \text{(Divide both numerator and denominator by 10.)} \\ &= \frac{0.2}{1-0.7}\end{aligned}$$

Compare this expression directly with $\frac{a}{1-r}$. Therefore, $a=0.2$ or $\frac{1}{5}$ and $r=0.7$ or $\frac{7}{10}$. Hence, $\frac{2}{3}=\frac{1}{5}+\frac{7}{50}+\frac{49}{500}+\ldots$. However, dividing both numerator and denominator by 10 was a completely arbitrary choice. What do you obtain when you divide by 4? by 2?

$$\begin{aligned}\frac{2}{3} &= \frac{\frac{2}{4}}{\frac{3}{4}} \quad \text{(Divide both numerator and denominator by 4.)} \\ &= \frac{\frac{1}{2}}{1-\frac{1}{4}} \\ &= \frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\ldots\end{aligned}$$

$$\frac{2}{3} = \frac{\frac{2}{2}}{\frac{3}{2}} \quad \text{(Divide both numerator and denominator by 2.)}$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)}$$

$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

Do you see that there is unlimited number of possibilities?

1. Express $\frac{1}{2}$ as an infinite series in three different ways.

2. Express 1 as an infinite series in three different ways. Do **not** simply multiply the answers to question 1 by two.

Check your answers by turning to the Appendix.

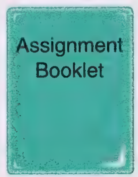
Infinite series may be used to represent a variety of everyday situations, such as the distance a ball bounces before coming to rest. But does a ball bounce indefinitely? When does it come to rest?

Conclusion

In this section, you were introduced to the concept of a limit, and the notation used in the context of infinite sequences and infinite geometric series. You should be able to explain what convergent infinite sequences and series are, describe conditions under which convergences occur, and calculate the limits of convergent sequences and infinite geometric series. You may wish to review some of the basics by reworking the examples in the activities.

The weathered hills and gullies in the Badlands reveal the geological past of that region. Each layer or stratum of earth and rock represents a different period of time. The upper layers represent more recent periods, whereas the lower layers represent a more distant past. When geologists or paleontologists look at strata, do they view them as a sequence or a series? If you are interested, you may wish to look up how geological time is sequenced and what periods are represented by the strata exposed in the Badlands.

Assignment



You are now ready to complete the section assignment.

Section 2: The Limit of a Function

NASA

In the past, people lived as though Earth's resources were boundless. However, if you view photographs of Earth taken from space, it becomes much more apparent just how limited the resources of the planet are. Water, air, and arable land are all finite. For example, scientists maintain that the ocean fishery is sustainable only within specific, well-defined limits. Stewardship of these resources is essential if future generations are to benefit.

The reference to limits is common in everyday conversation. How is the concept of limit in mathematics similar? How is it different?

In this section you will be introduced to the concept of limits within the context of algebraic functions. You will be working with functions with limits, with left-hand or right-hand limits, or with no limits.

You will determine the limit of any algebraic function as the independent variable approaches finite or infinite values, both for continuous and discontinuous functions. You will study the limit theorems for sums, differences, multiples, products, quotients, powers, and roots, citing suitable examples. You will then apply those limit theorems to determine numerical limits and, once found, to use those limits to analyse graphs.

Activity 1: Limit of a Function

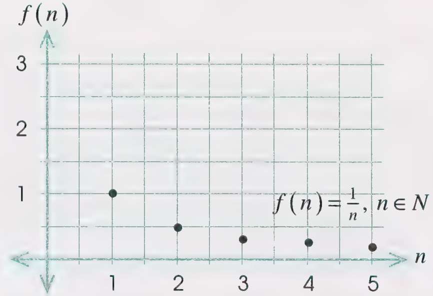
In the previous section, you found limits of sequences such as $f(n)=\frac{2}{n}$, where n was a natural number. But what about limits of functions such as $f(x)=\frac{1}{x}$, where x is a real number?

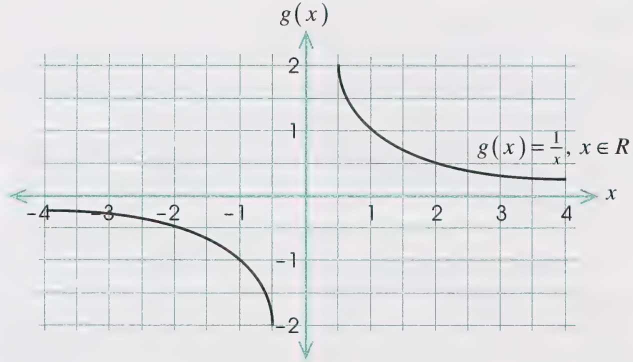
Functions, where the domains are the real numbers or subsets of the set of real numbers, are called **functions of a real variable**. Functions of a real variable such as $f(x)=2x^3-x$, $y=\frac{\sqrt{x-2}}{x^2-4}$, and $g(x)=\frac{x-2}{x+1}$ are called **algebraic functions**. An algebraic function is one which can be generated by combinations of the **algebraic operations** alone: addition and subtraction, multiplication and division, and finding powers and taking roots.

Example 1

Compare the graph of $f(n)=\frac{1}{n}$, where $n \in N$, with $g(x)=\frac{1}{x}$, where $x \in R$.

Solution





The graph of $f(n)=\frac{1}{n}$, where $n \in N$, consists of discrete points since the function is only defined for natural values of n. The graph of $g(x)=\frac{1}{x}$ is defined for all real values of x, except 0; therefore, the graph of $g(x)=\frac{1}{x}$ is continuous except at $x=0$.

Now, $\lim\limits_{n \to \infty} f(n) = 0$. Similarly, $g(x)$ approaches 0 as x increases without bound. For instance, $g(1000) = 0.001$.

$$\lim_{x \to \infty} g(x) = 0$$

If x decreases without bound, that is $x \to -\infty$, $g(x)$ approaches 0. For instance, $g(-1000) = -0.001$.

$$\lim_{x \to -\infty} g(x) = 0$$

What happens to the graph of $g(x) = \frac{1}{x}$ as x approaches 0 from the right? If you substitute progressively smaller values of x (such as 0.1, 0.01, and 0.001), $g(x)$ grows larger without bound.

x	0.1	0.01	0.001
$g(x)$	10	100	1000

As $x \to 0$ from the right, $g(x) \to \infty$. Notice as $x \to 0$ from the left, $g(x) \to -\infty$. For instance,

x	-0.1	-0.01	-0.001
$g(x)$	-10	-100	-1000

As you can see, the analysis of the graphs of functions of a real variable involves a study of limits.

Example 2

Determine what happens to the function $f(x) = \frac{x^2-16}{x-4}$ as x approaches 4.

Solution

For this rational algebraic function, $f(4)$ is not defined since $f(4) = \frac{0}{0}$. But what happens as x becomes arbitrarily close to 4? In particular, what happens when x approaches 4 from the right?

x	4.1	4.01	4.001	4.0001
$f(x)$	8.1	8.01	8.001	8.0001

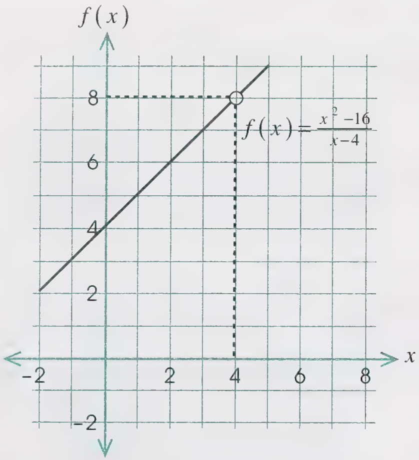
What happens as x approaches 4 from the left?

x	3.9	3.99	3.999	3.9999
$f(x)$	7.9	7.99	7.999	7.9999

In both situations, as x approaches 4, $f(x)$ approaches 8.

This fact is written $\lim\limits_{x \to 4} \frac{x^2-16}{x-4}$ and is read as **the limit as x approaches 4 is 8**.

This result is obvious if you graph $y = f(x)$.



The open circle at $(4, 8)$ is an essential element of the graph. It indicates that $(4, 8)$ is excluded from the graph since the function is undefined when $x = 4$.

The limit L of a function $y = f(x)$, when x approaches a from either side, is that value to which the function can be made to be arbitrarily close (whenever x is sufficiently close to a).

You will write $\lim_{x \to a} f(x) = L$.

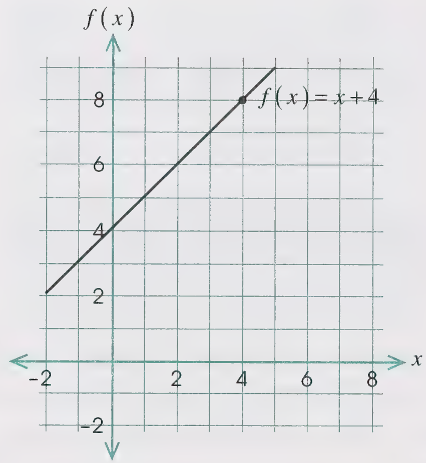
Example 3

Given $f(x) = x + 4$, find $\lim_{x \to 4} f(x)$.

Solution

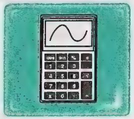
The graph of this function is identical to the graph in Example 2, except the function is defined when $x = 4$.

Here, $f(4) = 8$.



$$\therefore \lim_{x \to 4} f(x) = f(4)$$
$$= 8$$

Example 4



Graph $f(x)=\frac{x^3-8}{x-2}$ on a graphing calculator.

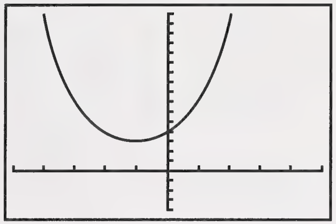
Determine $\lim_{x\to 2} f(x)$ by using the Trace feature.

Explain your result.

Solution

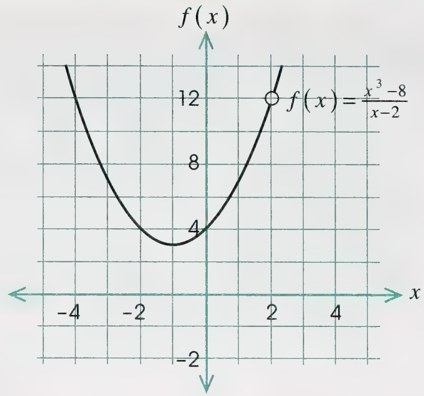
Graph (X,θ,T x^y 3 − 8)

÷ (X,θ,T − 2) EXE



Most graphing calculators display the graph of this function as a continuous curve. However, because $f(2)=\frac{0}{0}$ is meaningless, this function is not defined at $x=2$. Indicate this missing point by drawing an open circle on the graph where $x=2$.

The graph should look as follows:



Typical values displayed by a calculator near $x=2$ are

$x=1.808\,5106$	$y=10.897\,732$
$x=1.914\,8936$	$y=11.496\,604$
$x=2.021\,2765$	$y=12.128\,112$
$x=2.127\,6535$	$y=12.782\,254$

The Zoom feature may be used to come closer to $x=2$.

$x=1.971\,4803$	$y=11.829\,695$
$x=1.986\,1928$	$y=11.917\,347$
$x=2.000\,9053$	$y=12.005\,433$

It is therefore likely that $\lim_{x \to 2} f(x) = 12$. But how can you prove this algebraically?

What is equivalent to $\frac{x^3-8}{x-2}$? Factor the numerator using the pattern:

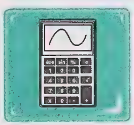
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\begin{aligned}\frac{x^3-8}{x-2} &= \frac{(x-2)(x^2+2x+4)}{x-2} \\ &= x^2+2x+4 \quad (\text{provided } x \neq 2)\end{aligned}$$

The graph of $y = x^2 + 2x + 4$ is identical to the preceding graph except it is defined for $x = 2$.

$$\begin{aligned}\text{At } x = 2,\ y &= 2^2 + 2(2) + 4 \\ &= 12\end{aligned}$$

Therefore, as $x \to 2$, the original function must approach 12. The point $(2, 12)$ is the point missing from the graph.



Use a graphing calculator or a computer graphing software (such as Zap-a-Graph™) to graph and find the limits of the following functions. Model Example 4 to find the limits algebraically.

1. $f(x) = \frac{x^2-1}{x-1}$, where $x \to 1$

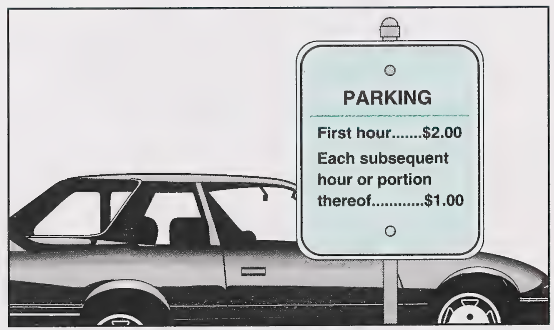
2. $f(x) = \frac{x^3+8}{x+2}$, where $x \to -2$

3. $f(x) = \frac{x^3-27}{x^2-9}$, where $x \to 3$

Check your answers by turning to the Appendix.

You may wish to review the similarities and differences between limits of sequences and limits of algebraic functions.

Activity 2: Left- and Right-Hand Limits

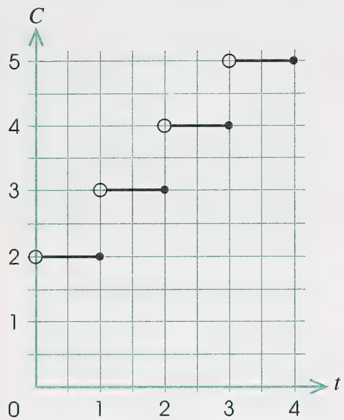


Have you ever parked in a commercial parking lot intending to stay just an hour, but when you returned just after the hour expired, discovered that you were charged for two hours? What do parking charges and finding limits in calculus have in common?

Example 1

A parking lot charges \$2.00 for the first hour and \$1.00 for every subsequent hour. Draw a graph of the charges C (in dollars) versus time t (in hours). Does the limit as $t \to 2$ h exist? Explain.

Solution



The graph is an example of a **step function**. Formally, a step function is a function which is constant throughout each interval for a set of non-intersecting intervals. For instance, in this example, the parking charges for $1 < t \leq 2$ is \$3.00, but steps up a dollar to \$4.00 for $2 < t \leq 3$, and so on.

As you can see from the graph, as you approach 2 h from the left, the charges remain constant at \$3.00. This limit, obtained by approaching $t = 2$ from the left, is called a **left-hand limit**. It is denoted as $\lim\limits_{t \to 2^-} f(t) = 3$. The raised negative sign indicates that 2 is approached from the left.

If you were to approach 2 h from the right, the charges are \$4.00, regardless of how short the time beyond 2 h might be. This limit, obtained by approaching $t = 2$ from the right, is called the **right-hand limit**. It is denoted as $\lim\limits_{t \to 2^+} f(t) = 4$. The raised positive sign indicates that 2 is approached from the right.

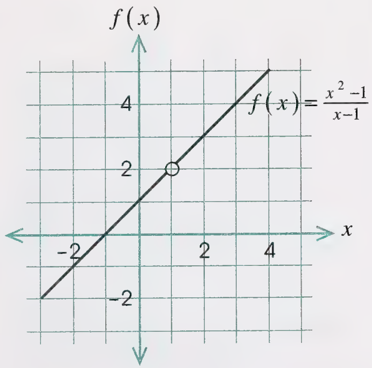
Because the left- and right-hand limits are not equal (you do not get the same value when you approach 2 from both sides), $\lim\limits_{t \to 2} f(t)$ does not exist. For the limit to exist, the left- and right-hand limits must be equal.

In general, for any function $y = f(x)$, if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$, then $\lim\limits_{x \to a} f(x) = L$. If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x)$ is undefined.

Example 2

Determine $\lim_{x\to 1^-} \frac{x^2-1}{x-1}$, $\lim_{x\to 1^+} \frac{x^2-1}{x-1}$, and $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Solution



$\lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2$ and $\lim_{x\to 1^+} \frac{x^2-1}{x-1} = 2$. Therefore, $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

Recall, from Activity 1

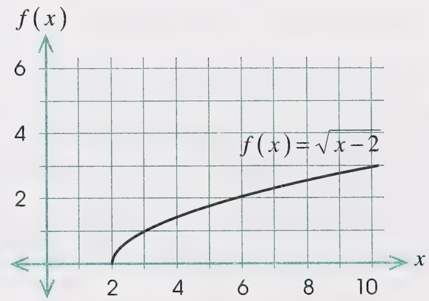
$$\begin{aligned}\lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x\to 1}(x+1) \\ &= 2\end{aligned}$$

Remember, you **are** allowed to cancel the common factor $x-1$, since $x \neq 1$. You are finding a limit as x approaches 1; therefore, x differs from 1, even though ever so slightly.

Example 3

Determine $\lim_{x\to 2^-} \sqrt{x-2}$, $\lim_{x\to 2^+} \sqrt{x-2}$, and $\lim_{x\to 2} \sqrt{x-2}$.

Solution



Since the domain of the function is $x \geq 2$, $\lim_{x\to 2^-} \sqrt{x-2}$ does not exist.

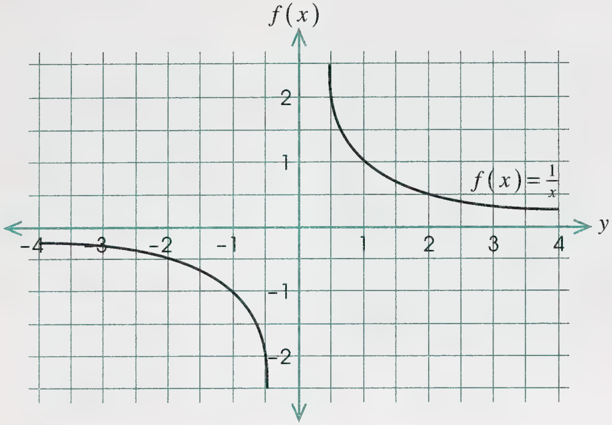
$$\lim_{x\to 2^+} \sqrt{x-2} = 0$$

Therefore, $\lim_{x\to 2} \sqrt{x-2}$ does not exist.

Example 4

Investigate what happens near $x=0$ for $f(x)=\frac{1}{x}$.

Solution



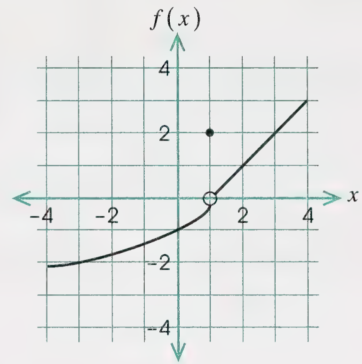
The graph of $f(x)=\frac{1}{x}$ is a continuous except at $x=0$.

What happens to the graph of $f(x)=\frac{1}{x}$ as x approaches 0 from the right? If you substitute progressively smaller values of x (such as 0.1, 0.01, and 0.001), $f(x)$ grows larger without bound. Therefore, $\lim\limits_{x\to 0^+}\frac{1}{x}=+\infty$. However, $\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$. Therefore, $\lim\limits_{x\to 0}\frac{1}{x}$ does not exist.

Example 5

Draw the graph of $f(x)=x-1$, when $x>1$, $f(x)=2$, when $x=1$, and $f(x)=-\sqrt{1-x}$, when $x<1$. Then find $\lim\limits_{x\to 1^-}f(x)$, $\lim\limits_{x\to 1^+}f(x)$, and $\lim\limits_{x\to 1}f(x)$.

Solution



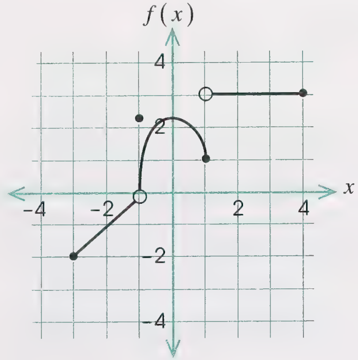
$$\lim_{x\to 1^-}f(x)=-\sqrt{1-x}$$
$$=0$$

$$\lim_{x\to 1^+}f(x)=x-1$$
$$=0$$

$$\therefore \lim_{x\to 1}f(x)=0$$

Note: The limit is 0 as x approaches 1 even though $f(1)=2$.

1. Using the graph $y = f(x)$, determine each limit.



a. $\lim_{x \to -1^-} f(x)$

b. $\lim_{x \to -1^+} f(x)$

c. $\lim_{x \to -1} f(x)$

d. $\lim_{x \to -3^-} f(x)$

e. $\lim_{x \to -3^+} f(x)$

f. $\lim_{x \to -3} f(x)$

g. $\lim_{x \to 1^-} f(x)$

h. $\lim_{x \to 1^+} f(x)$

i. $\lim_{x \to 1} f(x)$

j. $\lim_{x \to 3} f(x)$

2. Draw a graph of each function. Determine the limits if they exist.

a. $f(x) = |x| + 2$, where $\lim_{x \to 0} f(x)$

b. $f(x) = \sqrt{x+1} - 2$, where $\lim_{x \to -1^+} f(x)$

c. $f(x) = \dfrac{1}{x-2}$, where $\lim_{x \to 2^-} f(x)$

d. $f(x) = x^2 + x$, where $\lim_{x \to -1} f(x)$

e. $f(x) = -\sqrt{x}$, where $\lim_{x \to 0^-} f(x)$

f. $f(x) = -|x-1|$, where $\lim_{x \to 1^+} f(x)$

3. Draw the graph of $y = f(x)$ if it is defined as follows:

- $f(x) = -x + 1$ when $x < 0$
- $f(x) = -3$ when $x = 0$
- $f(x) = -x + 1$ when $x > 0$

Then, find these limits.

a. $\lim_{x \to 0^-} f(x)$

b. $\lim_{x \to 0^+} f(x)$

c. $\lim_{x \to 0} f(x)$

Check your answers by turning to the Appendix.

At the beginning of this activity you investigated a step function that modelled parking charges. Can you think of other everyday situations that might be represented by step functions?

Activity 3: Continuity

Fireworks create spectacular patterns. Showering sparks leave continuous trails of light and colour. The overall effect is a geometric exhibition of interrupted light and darkness.

Like those trails in the night sky, the graphs of algebraic functions can be continuous, or there may be breaks in those graphs.

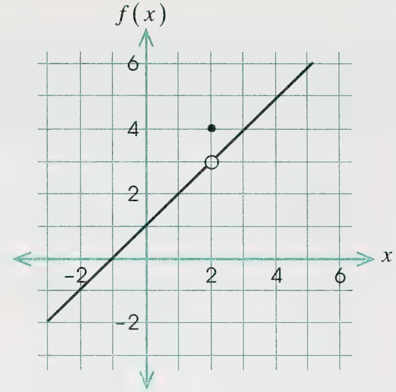
Example 1

Graph $f(x)$ if it is defined as follows:

- $f(x)=\dfrac{x^2-x-2}{x-2}$ when $x \neq 2$
- $f(x)=4$ when $x=2$

Why is this function not continuous at $x=2$? How can the equation be changed so that the resulting graph is continuous?

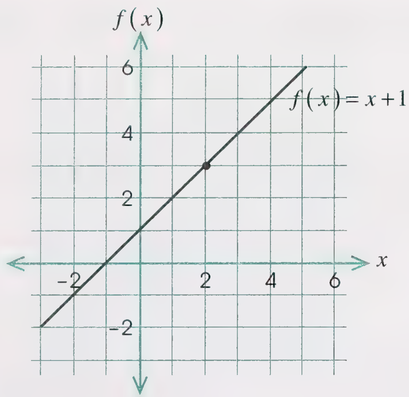
Solution



$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2}\frac{\cancel{(x-2)}(x+1)}{\cancel{x-2}}\\ &= \lim_{x\to 2}(x+1)\\ &= 3\end{aligned}$$

Since $x \neq 2$, simplify by dividing by $x-2$.

Since $f(2)=4$, $\lim f(x) \neq f(2)$. Therefore, the function is not continuous at $x=2$. To make the graph continuous at $x=2$, remove the point $(2,4)$ and insert the point $(2,3)$. That is, $\lim\limits_{x\to 2} f(x)=f(2)=3$ in order for it to be continuous.



This is simply the graph of $f(x) = x + 1$. For this function, since it is continuous, $\lim_{x \to a} f(x)$ equals $f(a)$ for any value of a.

In general, a function $y = f(x)$ is **continuous** at $x = a$ if $f(a)$ exists and $\lim_{x \to a} f(x) = f(a)$. A function that is not continuous at $x = a$ is said to be **discontinuous** at $x = a$, or that there is a discontinuity at $x = a$.

1. Go back to Examples 1 to 5 in Activity 2 from this section. Explain where and why the discontinuities of each function occur.

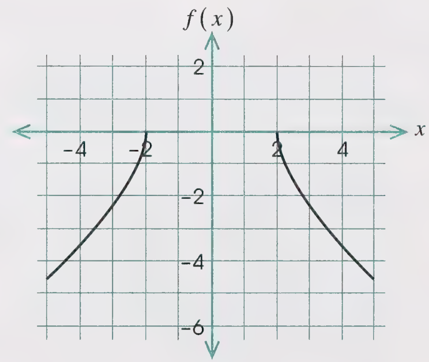
2. $f(x) = x^2 - 1$ is a continuous function. Explain how to find $\lim_{x \to 3} f(x)$.

Check your answers by turning to the Appendix.

Example 2

Where is $f(x) = -\sqrt{x^2 - 4}$ discontinuous? Why?

Solution



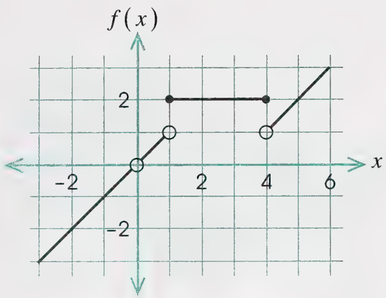
This function is discontinuous in the interval $[-2, 2]$, because $f(x)$ is not defined in $(-2, 2)$ and the limits as x approaches -2 or $+2$ do not exist.

Example 3

Discuss the discontinuities of the following function.

- $f(x) = \frac{x^2}{x}$ when $x < 1$
- $f(x) = 2$ when $1 \le x \le 4$
- $f(x) = x - 3$ when $x > 4$

Solution



The discontinuities occur at $x = 0$, because $f(0)$ is undefined; at $x = 1$, because $\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$; and at $x = 4$, because $\lim_{x \to 4^-} f(x) \ne \lim_{x \to 4^+} f(x)$.

3. To post a letter within Canada in 1994, the following rates were applied if the letter was enclosed in a standard-sized envelope.

Mass (g) up to and including	30	50	100	200	500
Rate (cents)	43	69	88	140	200

Graph the rate R (in cents) as a step function of the masses m (in grams). Discuss the discontinuities of the graph.

4. Sketch each function, and determine where discontinuities, if any, occur.

a. $f(x) = \sqrt{x^2 - 1}$

b. $f(x) = \frac{1}{x - 3}$

c. $f(x) = x^2 - 3x$

d. $f(x) = \frac{1}{x}$ when $x \ne 0$
$f(x) = 0$ when $x = 0$

e. $f(x) = \frac{x^2 - 1}{x + 1}$ when $x \ne -1$
$f(x) = 3$ when $x = -1$

Check your answers by turning to the Appendix.

Remember: Even though a function may contain discontinuities, it may be continuous for a given subset of its domain.

Activity 4: Limit Theorems

Have you ever wondered what the world would be like without rules and procedures? In mathematics, **theorems** are like those rules. They help give structure to the discipline. A theorem is a general statement or conclusion that is proven on the basis of certain given assumptions, definitions, and previously proven results.

In this section you have used limits rather informally, without a strict formalization of how the limits of algebraic functions are obtained. The following theorems summarize limit properties which you have already observed as you have worked through this section. The proofs of these theorems are beyond the scope of this course; however, if you pursue mathematics, the proofs will provide an important basis for further study in calculus.

In each theorem that follows, assume that the limits of functions exist and are finite.

Theorem 1: $\lim\limits_{x \to a} c = c$, where c is a constant

The limit of a constant is the constant itself.

Theorem 2: $\lim\limits_{x \to a} \left[f(x) + g(x) \right] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

The limit of a sum is the sum of the limits.

Theorem 3: $\lim\limits_{x \to a} \left[f(x) - g(x) \right] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

The limit of a difference is the difference of the limits.

Theorem 4: $\lim\limits_{x \to a} c f(x) = c \lim\limits_{x \to a} f(x)$, where c is a constant

The limit of a product of a constant and a function is the product of the constant and the limit of the function.

Theorem 5: $\lim\limits_{x \to a} \left[f(x) \bullet g(x) \right] = \lim\limits_{x \to a} f(x) \bullet \lim\limits_{x \to a} g(x)$

The limit of a product is the product of the limits.

Theorem 6: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, where $\lim\limits_{x \to a} g(x) \neq 0$

The limit of a quotient is the quotient of the limits.

Theorem 7: $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$, where $n \in N$

The limit of a power is the power of the limit.

Theorem 8: $\lim\limits_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$

The limit of a root is the root of the limit.

Example 1

Use the limit theorems to find $\lim\limits_{x\to 2}(2x-3)$.

Solution

$$\begin{aligned}\lim_{x\to 2}(2x-3) &= \lim_{x\to 2} 2x - \lim_{x\to 2} 3\\ &= 2\lim_{x\to 2} x - 3\\ &= 2 \bullet 2 - 3\\ &= 1\end{aligned}$$

The usual solution is a shortcut of this procedure.

$$\begin{aligned}\lim_{x\to 2}(2x-3) &= 2(2)-3\\ &= 1\end{aligned}$$

Example 2

Use the limit of theorems to find $\lim\limits_{x\to 4}\left(\sqrt{x}-3\right)^2$.

Solution

$$\begin{aligned}\lim_{x\to 4}\left(\sqrt{x}-3\right)^2 &= \left[\lim_{x\to 4}\left(\sqrt{x}-3\right)\right]^2\\ &= \left(\lim_{x\to 4}\sqrt{x} - \lim_{x\to 4} 3\right)^2\\ &= \left(\sqrt{\lim_{x\to 4} x} - 3\right)^2\\ &= \left(\sqrt{4}-3\right)^2\\ &= (-1)^2\\ &= 1\end{aligned}$$

Again, the usual procedure is shorter.

$$\begin{aligned}\lim_{x\to 4}\left(\sqrt{x}-3\right)^2 &= \left(\sqrt{4}-3\right)^2\\ &= (-1)^2\\ &= 1\end{aligned}$$

The application of the limit theorems is implicit.

Now it is your turn.

1. Apply the limit theorems to evaluate the following limits.

   a. $\lim\limits_{x\to 2}\left(5x^2-3\right)$

   b. $\lim\limits_{x\to 0}\sqrt{5x+4}$

Check your answers by turning to the Appendix.

Even though you have not proven the limit theorems, it is easy to verify them in a given situation.

Example 3

Verify that $\lim\limits_{x\to 3} 5(x+7) = 5\lim\limits_{x\to 3}(x+7)$.

Solution

LS	RS
$\lim\limits_{x\to 3} 5(x+7)$	$5\lim\limits_{x\to 3}(x+7)$
$=\lim\limits_{x\to 3} 5x+35$	$=5(3+7)$
$=5(3)+35$	$=5\times 10$
$=50$	$=50$

LS = RS

Example 4

Verify that $\lim\limits_{x\to 2}\left[(2x+1)+(3x-4)\right]=\lim\limits_{x\to 2}(2x+1)+\lim\limits_{x\to 2}(3x-4)$.

Solution

LS	RS
$\lim\limits_{x\to 2}\left[(2x+1)+(3x-4)\right]$	$\lim\limits_{x\to 2}(2x+1)+\lim\limits_{x\to 2}(3x-4)$
$=\lim\limits_{x\to 2}\left[5x-3\right]$	$=2(2)+1+2(3)-4$
$=5(2)-3$	$=4+1+6-4$
$=10-3$	$=7$
$=7$	

LS = RS

Example 5

Verify that $\lim\limits_{x\to 1}\left[(x-3)(x+2)\right]=\lim\limits_{x\to 1}(x-3)\bullet\lim\limits_{x\to 1}(x+2)$.

Solution

LS	RS
$\lim\limits_{x\to 1}\left[(x-3)(x+2)\right]$	$\lim\limits_{x\to 1}(x-3)\bullet\lim\limits_{x\to 1}(x+2)$
$=\lim\limits_{x\to 1}\left[x^2-x-6\right]$	$=(1-3)(1+2)$
$=1-1-6$	$=(-2)(3)$
$=-6$	$=-6$

LS = RS

Example 6

Verify that $\lim\limits_{x\to 3}\left(\frac{x-1}{x+2}+\frac{x^2-9}{x-3}\right)=\lim\limits_{x\to 3}\frac{x-1}{x+2}+\lim\limits_{x\to 3}\frac{x^2-9}{x-3}$.

Solution

LS	RS
$\lim\limits_{x\to 3}\left(\dfrac{x-1}{x+2}+\dfrac{x^2-9}{x-3}\right)$	$\lim\limits_{x\to 3}\dfrac{x-1}{x+2}+\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}$
$=\lim\limits_{x\to 3}\left[\dfrac{(x-1)}{(x+2)}+\dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}\right]$	$=\lim\limits_{x\to 3}\dfrac{x-1}{x+2}+\lim\limits_{x\to 3}\dfrac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$
$=\lim\limits_{x\to 3}\left[\dfrac{x-1}{x+2}+x+3\right]$	$=\dfrac{3-1}{3+2}+(3+3)$
$=\dfrac{3-1}{3+2}+3+3$	$=\dfrac{2}{5}+6$
$=6\dfrac{2}{5}$	$=6\dfrac{2}{5}$

LS = RS

2. Verify the following.

a. $\lim_{x\to5}\left[\left(\frac{x^2-25}{x+5}\right)\left(\frac{x^2-25}{x-5}\right)\right]=\lim_{x\to5}\frac{x^2-25}{x+5}\bullet\lim_{x\to5}\frac{x^2-25}{x-5}$

b. $\lim_{x\to3}\frac{\left(\frac{x^2-9}{x+3}\right)}{\left(\frac{x^2-9}{x-3}\right)}=\frac{\lim_{x\to3}\left(\frac{x^2-9}{x+3}\right)}{\lim_{x\to3}\left(\frac{x^2-9}{x-3}\right)}$

c. $\lim_{x\to2}(2)\left(\frac{x^2-4}{x-2}\right)=2\lim_{x\to2}\frac{x^2-4}{x-2}$

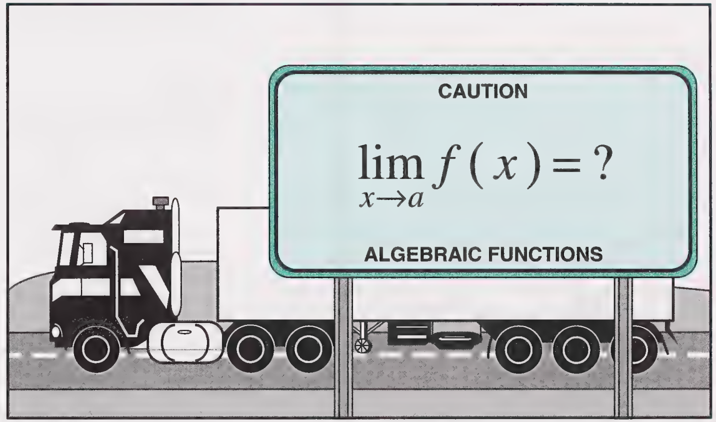
d. $\lim_{x\to1}\left(\frac{x^2-4x+3}{x-1}-\frac{x^2+x-2}{x-1}\right)$

$$=\lim_{x\to1}\frac{x^2-4x+3}{x-1}-\lim_{x\to1}\frac{x^2+x-2}{x-1}$$

Check your answers by turning to the Appendix.

You may wish to make your own rule book, listing limit theorems and other procedures from this module.

Activity 5: Limits of Algebraic Functions



You are now ready to investigate techniques for finding limits of algebraic functions.

Example 1

Determine $\lim_{x\to2}\left(5x^2-3x-2\right)$.

Solution

Because $f(x)=5x^2-3x-2$ is a continuous function, $\lim_{x\to2}f(x)=f(2)$.

$$\therefore \lim_{x \to 2} f(x) = 5(2)^2 - 3(2) - 2$$
$$= 20 - 6 - 2$$
$$= 12$$

Example 2

Given $f(x) = \frac{4 - x^2}{x - 2}$, find $\lim_{x \to 3} f(x)$.

Solution

This function is only discontinuous at $x = 2$.

$$\therefore \lim_{x \to 3} f(x) = f(3)$$
$$= \frac{4 - 3^2}{3 - 2}$$
$$= -5$$

Example 3

Given $f(x) = \frac{4 - x^2}{x - 2}$, find $\lim_{x \to 2} f(x)$.

Solution

This function is discontinuous at 2. Since $f(2) = \frac{0}{0}$ is meaningless, you must simplify to obtain a function that is continuous at 2.

$$\lim_{x \to 2} \frac{4 - x^2}{x - 2} = \lim_{x \to 2} \frac{(2 - x)(2 + x)}{x - 2}$$
$$= \lim_{x \to 2} \frac{-\cancel{(-2 + x)}(2 + x)}{\cancel{(-2 + x)}}$$
$$= \lim_{x \to 2} [-(2 + x)]$$
$$= -4$$

You can reduce since x is approaching 2.

1. Give an example of a polynomial function for which $\lim_{x \to 0} f(x) = 0$. What does that tell you about its graph?

2. Give an example of a rational algebraic function, that is $f(x) = \frac{g(x)}{h(x)}$, for which the limit as $x \to a$ is $f(a)$ for any real value of a.

Check your answers by turning to the Appendix.

Example 4

Evaluate $\lim_{x \to 4} \frac{x + 2}{x - 4}$.

Solution

You cannot replace this function by another which is continuous at $x = 4$.

Therefore, $\lim_{x \to 4} \frac{x+2}{x-4}$ does not exist.

Before attempting the following questions, watch the video *An Introduction to Calculus and Vectors* from the *Catch 31* series, ACCESS Network. In particular, review the techniques for finding limits. This video is available from Learning Resources Distributing Centre.

3. Find the limits of L in each of the following.

a. $L = \lim_{x \to -3} \frac{x^2 - 9}{x + 3}$

b. $L = \lim_{x \to 1} \frac{x^2 + 7x - 8}{x - 1}$

c. $L = \lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3}$

d. $L = \lim_{x \to 2} \frac{5x - 10}{x - 2}$

e. $L = \lim_{x \to 1} \frac{6x - 2}{3x - 1}$

f. $L = \lim_{x \to 1} \frac{x - 1}{x^2 - 1}$

g. $L = \lim_{x \to 2} \frac{x + 2}{x - 3}$

h. $L = \lim_{x \to 5} \frac{x^2 - x - 2}{x - 5}$

i. $L = \lim_{x \to 9} \frac{x^2 - 1}{x - 9}$

j. $L = \lim_{x \to 0} \frac{\left(3 - \frac{1}{x}\right)}{\left(\frac{2}{x} - 5\right)}$

k. $L = \lim_{x \to 3} \frac{x^3 - 27}{x - 3}$

Check your answers by turning to the Appendix.

Example 5

Determine $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$.

Solution

$$
\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} &= \lim_{x \to 0} \frac{\left(\sqrt{x+4} - 2\right)\left(\sqrt{x+4} + 2\right)}{x\left(\sqrt{x+4} + 2\right)} \\
&= \lim_{x \to 0} \frac{\left(\sqrt{x+4}\right)^2 - 2^2}{x\left(\sqrt{x+4} + 2\right)} \\
&= \lim_{x \to 0} \frac{(x + 4 - 4)}{x\left(\sqrt{x+4} + 2\right)} \\
&= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x+4} + 2\right)} \\
&= \lim_{x \to 0} \frac{1}{\left(\sqrt{x+4} + 2\right)} \\
&= \frac{1}{\sqrt{0+4} + 2} \\
&= \frac{1}{4}
\end{aligned}
$$

Example 6

Evaluate $\lim_{h \to 0} \frac{(1-h)^2 - 1}{h}$.

Solution

$$\lim_{h \to 0} \frac{(1-h)^2 - 1}{h} = \lim_{h \to 0} \frac{\left(1 - 2h + h^2\right) - 1}{h}$$
$$= \lim_{h \to 0} \frac{-2h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(-2+h)}{\cancel{h}}$$
$$= -2 + 0$$
$$= -2$$

3. Evaluate $\lim_{h \to 0} \frac{(1-h)^3 - 1}{h}$.

4. Evaluate $\lim_{x \to 0} \frac{5 - \sqrt{25 - x}}{x}$.

Check your answers by turning to the Appendix.

Mathematics is a cumulative study. The topics you studied in Mathematics 20 are fundamental to your success with limits.

Activity 6: Limits of Infinity

If you have walked along a straight section of railway tracks, you may have noticed that the rails appear to converge in the distance. Artists use the principle of a vanishing point to represent landscapes realistically. The fence posts in the photograph give an illusion of distance as they grow smaller and smaller.

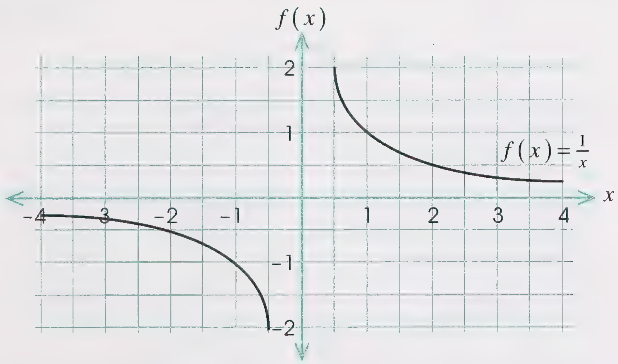
When graphing a function, it is often important that you know what happens to the graph as x increases without bound $(x \to \infty)$ or when x decreases without bound $(x \to -\infty)$.

You have already investigated these limits from the context of the function $f(x) = \frac{1}{x}$.

Example 1

Determine $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.

Solution



As x increases without bound, $f(x)$ approaches 0. For instance, $f(1\,000\,000)=0.000\,001$.

Therefore, $\lim_{x\to\infty} f(x)=0$.

As x decreases without bound, that is $x\to-\infty$, $f(x)$ approaches 0. For instance, $f(-1\,000\,000)=-0.000\,001$.

Therefore, $\lim_{x\to-\infty} f(x)=0$.

As $x\to\pm\infty$, the graph approaches but never intersects the x-axis. The x-axis, in this case, is a horizontal asymptote.

Determining asymptotes is an important skill in graphing certain functions. You will meet this topic again in Module 4.

Example 2

Determine $\lim_{x\to\infty}\frac{2x-1}{x-2}$.

Solution

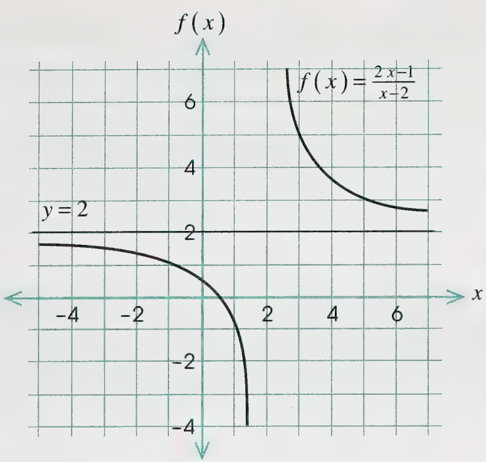
The process of finding a limit of an algebraic function as $x\to\infty$ parallels the process of finding the limit of a sequence as $n\to\infty$.

Begin by removing the highest power of the variable as a common factor.

$$\lim_{x\to\infty}\frac{2x-1}{x-2}=\lim_{x\to\infty}\frac{x\left(2-\frac{1}{x}\right)}{x\left(1-\frac{2}{x}\right)}$$

$$=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{1-\frac{2}{x}} \quad (\text{since } x\neq 0)$$

$$=2 \quad \left(\text{since } \tfrac{1}{x}\to 0 \text{ and } \tfrac{2}{x}\to 0 \text{ as } x\to\infty\right)$$



As you can see from this graph, the line $y = 2$ is a horizontal asymptote.

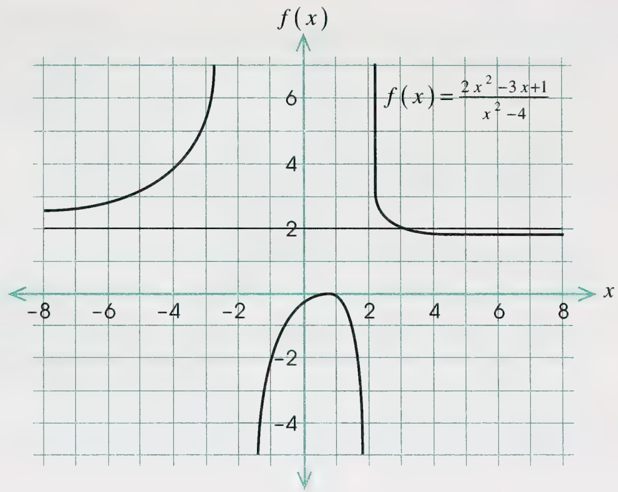
Example 3

Determine $\lim\limits_{x\to\infty} \dfrac{2x^2 - 3x + 1}{x^2 - 4}$.

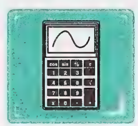
Solution

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 1}{x^2 - 4} = \lim_{x\to\infty} \frac{x^2\left(2 - \frac{3}{x} + \frac{1}{x^2}\right)}{x^2\left(1 - \frac{4}{x^2}\right)}$$

$$= 2$$



Again the horizontal asymptote is $y = 2$.

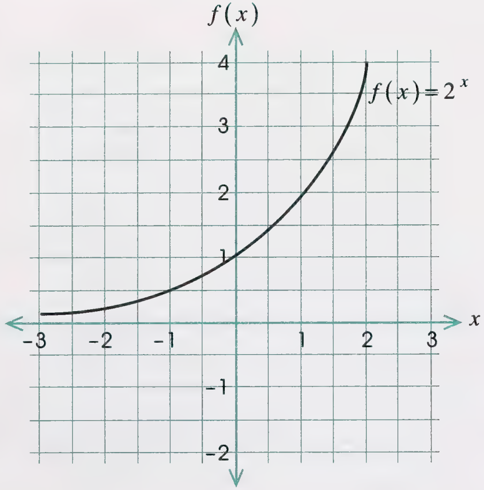


Confirm your results using a graphing calculator.

Example 4

Discuss $\lim\limits_{x\to+\infty} 2^x$ and $\lim\limits_{x\to-\infty} 2^x$ in terms of the graph of the function $f(x) = 2^x$.

Solution



As the graph indicates, when x increases without bound, 2^x increases without bound. Confirm this by using a table of values.

x	−3	−2	−1	0	1	2	3	4	5
$f(x)$	0.125	0.25	0.5	1	2	4	8	16	32

$$\therefore \lim_{x\to+\infty} 2^x = +\infty$$

As x decreases without bound, 2^x approaches 0.

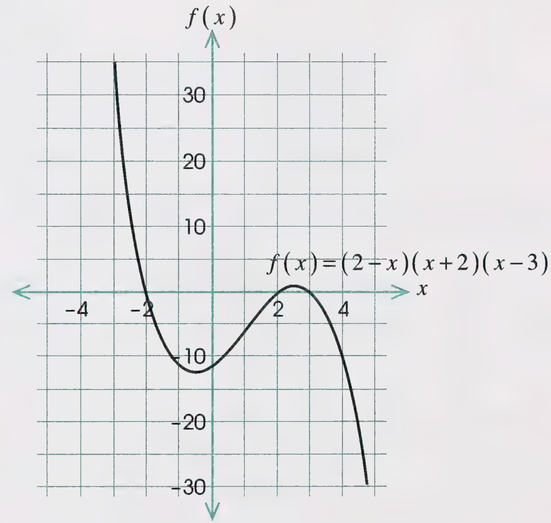
$$\therefore \lim_{x\to-\infty} 2^x = +0$$

The x-axis is a horizontal asymptote.

Example 5

Given $f(x)=(2-x)(x+2)(x-3)$, find $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution



If x increases without bound, $f(x)$ decreases without bound. When $x > 3$, the signs of the factors are $f(x) = (-)(+)(+)$. Therefore, $f(x)$ must be negative.

Since the absolute value of each factor increases as $x \to \infty$, $\lim_{x \to +\infty} f(x) = -\infty$.

If x decreases without bound, $f(x)$ increases without bound. When $x < -2$, the signs of the factors are $f(x) = (+)(-)(-)$. Therefore, $f(x)$ must be positive.

Since the absolute value of each factor increases as $x \to -\infty$, $\lim_{x \to -\infty} f(x) = \infty$.

There is no horizontal asymptote.

1. Evaluate each limit if it exists.

a. $\lim_{x \to \infty} \frac{2}{x-2}$

b. $\lim_{x \to -\infty} 3^{-x}$

c. $\lim_{x \to \infty} \frac{3x+2}{2x-1}$

d. $\lim_{x \to -\infty} \frac{x^2-2x+1}{2x^2-3x}$

e. $\lim_{x \to \infty} \frac{2x^2+3}{x+1}$

f. $\lim_{x \to -\infty} x(x-1)^2$

2. Give an example of a rational algebraic function with a horizontal asymptote of $y = -3$.

Use a graphing calculator to do question 3.

3. Draw the graph of $y = \frac{x^2}{(x-2)^2}$. Find the horizontal asymptote algebraically, and check your answer against the graph.

Check your answers by turning to the Appendix.

If you have time, investigate the mathematics of the vanishing point in art.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

To find limits such as $\lim_{x \to -2} \frac{x^3+8}{x^2-4}$ and $\lim_{x \to 2} \frac{x+5}{x^3-125}$, you must be able to factor the sum of two cubes, $a^3 + b^3$, and the difference of two cubes, $a^3 - b^3$. The patterns for factoring these expressions may be developed from numerical examples.

Example 1

Factor 2^3+3^3.

Solution

$$\begin{aligned}2^3+3^3&=8+27\\&=35\\&=(5)(7)\end{aligned}$$

How can these two factors be expressed in terms of the original values of 2 and 3?

Certainly 5 is obvious: $5=2+3$. But what about 7?

You must remember that the original expression was 2^3+3^3.

$$\begin{aligned}\therefore\ 2^3+3^3&=(5)(7)\\&=(2+3)\left(2^2+?+3^2\right)\end{aligned}$$

In the second factor, the first term 2^2 and the third term 3^2 total 13.

However, the second factor should only be 7. 13 is six too large!

$$\begin{aligned}\therefore\ 7&=2^2-6+3^2\\&=2^2-2(3)+3^2\end{aligned}$$

Putting everything together, $2^3+3^3=(2+3)\left(2^2-2(3)+3^2\right)$.

The results from Example 1 may be used to factor a^3+b^3. Simply replace 2 by a and 3 by b in $2^3+3^3=(2+3)\left(2^2-2(3)+3^2\right)$.

$$\therefore\ a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$

Check: By Multiplying

$$\begin{array}{r}a^2-ab+b^2\\ a+b\\ \hline a^2b-ab^2+b^3\\ a^3-a^2b+ab^2\qquad\\ \hline a^3\qquad\qquad\quad+b^3\end{array}$$

Many students forget how to get the middle term in the trinomial factor. Simply multiply the terms in the binomial factor, then change the sign of that product.

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$

Multiply a by b and change the sign.

The factors of a^3-b^3 may be obtained as follows.

$$\begin{aligned} a^3 - b^3 &= a^3 + (-b)^3 \\ &= [a + (-b)][a^2 - a(-b) + (-b)^2] \\ &= (a-b)(a^2 + ab + b^2) \end{aligned}$$

Again, the middle term in the trinomial factor is found by multiplying the terms in the binomial factor, then changing the sign of that product.

Example 2

Factor $x^3 + 8$.

Solution

Use the pattern $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

$$\begin{aligned} x^3 + 8 &= x^3 + 2^3 \\ &= (x+2)(x^2 - 2x + 2^2) \end{aligned}$$

Multiply x by 2 and change the sign.

Example 3

Factor $8x^3 - 27$.

Solution

Use the pattern $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$$\begin{aligned} 8x^3 - 27 &= (2x)^3 - 3^3 \\ &= (2x-3)((2x)^2 + 6x + (-3)^2) \\ &= (2x-3)(4x^2 + 6x + 9) \end{aligned}$$

Now try factoring on your own.

1. Factor $x^3 - 1$.

2. Factor $a^3b^3 + 1000$.

3. Factor $x^6 - y^3$.

4. Factor $x^6 - y^6$.

Check your answers by turning to the Appendix.

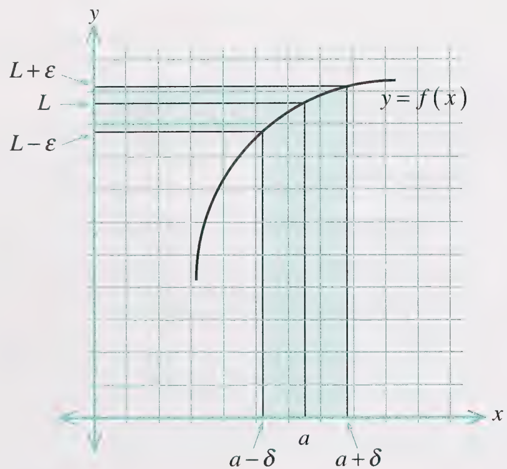
Enrichment

If you plan to take calculus courses beyond Mathematics 31, limits will be defined in more detail than they were in this module.

Remember that the limit of a function $y = f(x)$, as $x \to a$, is that value L from which the function differs by an arbitrarily small amount. This holds as long as x is sufficiently close to a.

More formally, the function $y = f(x)$ has the limit L as $x \to a$. That is, $\lim\limits_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, regardless how small, there corresponds a number $\delta > 0$, such that $|f(x) - L| < \varepsilon$ when $|x - a| < \delta$.

Consider the following graph.



$f(x)$ is within the interval $(L - \varepsilon, L + \varepsilon)$ when x is in the interval $(a - \delta, a + \delta)$.

Example 1

Given $f(x) = \frac{x^2 - 1}{x - 1}$, how close to 1 should you make x so that $\lim\limits_{x \to 1} f(x)$ differs from 2 by less than 0.001?

Solution

You must find x such that $|f(x) - 2| < 0.001$.

$$\left|\frac{x^2 - 1}{x - 1} - 2\right| < 0.001$$

$$\left|\frac{(x-1)(x+1)}{x - 1} - 2\right| < 0.001$$

$$|(x+1) - 2| < 0.001$$

$$|x - 1| < 0.001$$

$$-0.001 < x - 1 < 0.001$$

$$1 - 0.001 < x - 1 + 1 < 1 + 0.001$$

$$0.999 < x < 1.001$$

Thus, x must be in the interval $(0.999, 1.001)$ so that $f(x)$ lies within $(1.999, 2.001)$.

$$\therefore \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Can you show this algebraically?

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} \\ &= \lim_{x \to 1} x + 1 \\ &= 2 \end{aligned}$$

Example 2

For what value of x does $\frac{x^2-9}{x-3}$ differ from its limit by less than 10^{-5} as $x \to 3$?

Solution

$$
\begin{aligned}
L &= \lim_{x\to 3} \frac{x^2-9}{x-3} \\
&= \lim_{x\to 3} (x+3) \\
&= 6
\end{aligned}
$$

$\therefore \ |f(x)-L| < 10^{-5}$, where $x \neq 3$

$$\left|\frac{x^2-9}{x-3}-6\right| < 10^{-5}$$

$$|x+3-6| < 10^{-5}$$

$$|x-3| < 10^{-5}$$

$$-10^{-5} < x-3 < 10^{-5}$$

$$3-10^{-5} < x < 3+10^{-5}$$

Example 3

For what value of x does $(x-3)$ differ from its limit by less than 10^{-5} as $x \to 4$?

Solution

$$
\begin{aligned}
\lim_{x\to 4} (x-3) &= 4-3 \\
&= 1
\end{aligned}
$$

$$|f(x)-L| = |(x-3)-1|$$

If $|(x-3)-1| < 10^{-5}$, then $|x-4| < 10^{-5}$.

$\therefore \ 4-10^{-5} < x < 4+10^{-5}$

The same approach may be used with infinite sequences.

Example 4

For what values of n does $2+\frac{1}{n^2}$ differ from its limit by less than 10^{-4} as $n \to \infty$?

Solution

$$
\begin{aligned}
L &= \lim_{n\to\infty} \left(2+\frac{1}{n^2}\right) \\
&= 2
\end{aligned}
$$

$$|f(n)-L|<10^{-4}$$

$$\left|2+\frac{1}{n^2}-2\right|<10^{-4}$$

$$\left|\frac{1}{n^2}\right|<10^{-4}$$

$$\frac{1}{n^2}<10^{-4}$$

$$\frac{10^4}{n^2}<1 \quad \left(\text{since } 10^4 \bullet \tfrac{1}{n^2}<10^{-4} \bullet 10^4 \text{ and } 10^0=1\right)$$

$$10^4<n^2$$

$$n>10^2$$

$$>100$$

1. For what value of x does $(x+5)$ differ from its limit by less than 10^{-2} as $x \to 3$?

2. Find the number of terms such that $0.02+0.002+0.0002+\ldots$ differs from its limit L by less than 10^{-8} as the number of terms $n \to \infty$. Find $|S_n - L|$ first.

Check your answers by turning to the Appendix.

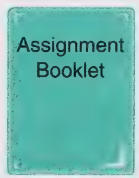
Conclusion

In this section, you explored the concept of a limit of a function. You investigated functions with limits, with left-hand or right-hand limits, or with no limits. As a result, you should now be able to determine the limit of any continuous or discontinuous algebraic function as the independent variable approaches finite or infinte values. Also, you should be able to use those limits, once found, to help you sketch both continuous and discontinuous functions.

You investigated bounded and unbounded functions, and bounded functions with no limits. Finally, you applied the limit theorems for sums, differences, multiples, products, quotients, powers, and roots.

Even though the concept of limit is precisely defined in mathematics; it is often used in much the same way in other contexts. In the fishery business for example, it is important that catch limits of a particular species be carefully regulated to maintain a critical number of breeding fish. If those limits are exceeded, the population of that species is likely to collapse. In a finite world, to preserve resourses for future generations, we must all be conscious of limits on human activity.

Assignment



You are now ready to complete the section assignment.

Module Summary

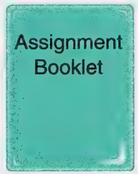
You have just completed Module 2. In this module you were introduced to a key concept in calculus – limits. In subsequent modules you will be using limits to develop both differential calculus and integral calculus. Within these branches, the development of notions such as instantaneous velocity and acceleration and formulas for areas and volumes of figures, like spheres or cones, will hinge on limits.

The coins and bills in the photograph have the same value regardless from what side you view or use them. In mathematics, numerous topics may be developed from differing viewpoints. However, in spite of the route taken, the destination will most often be the same. The various perspectives arise, in part, because of different ways concepts are applied, not only in mathematics itself, but also in real-world situations.

You studied limits from two perspectives: sequences and series on the one hand and algebraic functions on the other. You should now be able to discuss the unity of the concept of limit from both points of view.

In Module 3 you will be using limits to develop the first major section in calculus—the derivative. Good luck!

Final Module Assignment



You are now ready to complete the final module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

algebraic functions: functions which can be generated by the combination of algebraic operations alone

algebraic operations: addition, subtraction, multiplication, division, powers, and roots

asymptote: a line in which the graph of a function $y = f(x)$ approaches as x increases or decreases without bound

conjugate expressions: obtained by changing the sign (addition or subtraction) between the two terms of a given expression

For instance, the conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$, and vice versa.

continuous: at $x = a$, for the function f, when $f(a)$ is defined and

$\lim_{x \to a} f(x) = f(a)$

convergent sequence: an infinite sequence for which the terms approach a unique finite value

discontinuous: at $x = a$, for the function f, where the function is not continuous at the given value of x

divergent sequence: an infinite sequence that does not converge

functions of a real variable: functions where the domains are the set of real numbers or a subset of the real numbers

infinite geometric series: the sum of the terms of an infinite geometric sequence

infinite sequence: the range of a function which has the set of natural numbers as its domain

infinite series: the sum of the terms of an infinite sequence

left-hand limit: $\left(\lim_{x \to a^-} f(x) \right)$ the limiting value of a function f as x approaches a from the left

limit: the value L from which the function $y = f(x)$ differs by an arbitrarily small amount, as long as x is sufficiently close to a

right-hand limit: $\left(\lim_{x \to a^+} f(x) \right)$ the limiting value of a function f as x approaches a from the right

sequence: the range of a function which has the set of natural numbers as its domain or a subset of the natural numbers as its domain

series: the sum of the terms of a sequence

step function: a function which is constant throughout each interval for a set of non-intersecting intervals

theorem: a mathematical statement which requires proof

Suggested Answers

Section 1: Activity 1

1. a.

$$\begin{aligned} f(1) &= 5(1)-3 \\ &= 2 \end{aligned} \qquad \begin{aligned} f(2) &= 5(2)-3 \\ &= 7 \end{aligned}$$

$$\begin{aligned} f(3) &= 5(3)-3 \\ &= 12 \end{aligned} \qquad \begin{aligned} f(4) &= 5(4)-3 \\ &= 17 \end{aligned}$$

$$\begin{aligned} f(5) &= 5(5)-3 \\ &= 22 \end{aligned}$$

The first five terms are 2, 7, 12, 17, and 22.

b.

$$\begin{aligned} f(1) &= 2(1)^2+1 \\ &= 2+1 \\ &= 2 \end{aligned} \qquad \begin{aligned} f(2) &= 2(2)^2+1 \\ &= 8+1 \\ &= 9 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(3)^2+1 \\ &= 18+1 \\ &= 19 \end{aligned} \qquad \begin{aligned} f(4) &= 2(4)^2+1 \\ &= 32+1 \\ &= 33 \end{aligned}$$

$$\begin{aligned} f(5) &= 2(5)^2+1 \\ &= 50+1 \\ &= 51 \end{aligned}$$

The first five terms are 3, 9, 19, 33, and 51.

c.

$$\begin{aligned} t_1 &= 3+\frac{2}{1} \\ &= 3+2 \\ &= 5 \end{aligned} \qquad \begin{aligned} t_2 &= 3+\frac{2}{2} \\ &= 3+1 \\ &= 4 \end{aligned} \qquad \begin{aligned} t_3 &= 3+\frac{2}{3} \\ &= \frac{9}{3}+\frac{2}{3} \\ &= \frac{11}{3} \end{aligned}$$

$$\begin{aligned} t_4 &= 3+\frac{2}{4} \\ &= \frac{6}{2}+\frac{1}{2} \\ &= \frac{7}{2} \end{aligned} \qquad \begin{aligned} t_5 &= 3+\frac{2}{5} \\ &= \frac{15}{5}+\frac{2}{5} \\ &= \frac{17}{5} \end{aligned}$$

The first five terms are 5, 4, $\frac{11}{3}$, $\frac{7}{2}$, and $\frac{17}{5}$.

d.

$$\begin{aligned} t_1 &= 3^1+1 \\ &= 3+1 \\ &= 4 \end{aligned} \qquad \begin{aligned} t_2 &= 3^2+1 \\ &= 9+1 \\ &= 10 \end{aligned} \qquad \begin{aligned} t_3 &= 3^3+1 \\ &= 27+1 \\ &= 28 \end{aligned}$$

$$\begin{aligned} t_4 &= 3^4+1 \\ &= 81+1 \\ &= 82 \end{aligned} \qquad \begin{aligned} t_5 &= 3^5+1 \\ &= 243+1 \\ &= 244 \end{aligned}$$

The first five terms are 4, 10, 28, 82, and 244.

2. a.

$$\begin{aligned} f(3) &= -3^3 \\ &= -27 \end{aligned}$$

b.

$$\begin{aligned} t_8 &= 2^{-8}-1 \\ &= \frac{1}{256}-1 \\ &= -\frac{255}{256} \end{aligned}$$

c.

$$\begin{aligned} f(9) &= 3-8(9) \\ &= 3-72 \\ &= -69 \end{aligned}$$

3. **a.** The sequence is convergent.

$$\lim_{n\to\infty} \frac{6-n}{n-4} = -1$$

(6 − 1 0 0 0)

÷ (1 0 0 0 − 4) =

-0.99799 1967

$\therefore f(1000) \doteq -0.998$

If $n = 10\ 000$, then $f(10\ 000) \doteq -0.9998$.

b. The sequence is divergent. Therefore, there is no limit.

c. The sequence is convergent.

$$\lim_{x\to\infty} \frac{1}{n} = 0$$

1 ÷ 1 0 0 0 =

0.001

$\therefore f(1000) = 0.001$

If $n = 1\ 000\ 000$, then $f(1\ 000\ 000) = 0.000\ 001$

4. **a.**
$$\begin{aligned}\lim_{n\to\infty} \frac{n+3}{n} &= \lim_{n\to\infty}\left(\frac{n}{n}+\frac{3}{n}\right)\\ &= \lim_{n\to\infty}\left(1+\frac{3}{n}\right)\\ &= 1\end{aligned}$$

b.
$$\begin{aligned}\lim_{n\to\infty} \frac{5+3n}{n} &= \lim_{n\to\infty}\left(\frac{5}{n}+3\right)\\ &= 3\end{aligned}$$

c.
$$\begin{aligned}&\lim_{n\to\infty} \frac{3n^3-2n+1}{4n^3-2n^2+n-7}\\ &= \lim_{n\to\infty} \frac{3-\frac{2}{n^2}+\frac{1}{n^3}}{4-\frac{2}{n}+\frac{1}{n^2}-\frac{7}{n^3}}\\ &= \frac{3}{4}\end{aligned}$$

Divide both numerator and denominator by n^3.

d.
$$\begin{aligned}\lim_{n\to\infty} \frac{n^3-8n^2+3n+1}{n^3-2n+8} &= \lim_{n\to\infty} \frac{1-\frac{8}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{1-\frac{2}{n^2}+\frac{8}{n^3}}\\ &= 1\end{aligned}$$

e. $$\lim_{n\to\infty}\frac{n^2+2}{n^3-1}=\lim_{n\to\infty}\frac{n^3\left(\frac{1}{n}+\frac{2}{n^3}\right)}{n^3\left(1-\frac{1}{n^3}\right)}$$
$$=\frac{0+0}{1-0}$$
$$=0$$

f. $$\lim_{n\to\infty}\frac{n^3}{n^2+2}=\lim_{n\to\infty}\frac{n^3(1)}{n^3\left(\frac{1}{n}+\frac{2}{n^3}\right)}$$
$$=\frac{1}{0+0}$$
$$=\text{undefined}$$

5. If the degree of the numerator is less than the degree of the denominator, then the limit is 0. If the degree of the denominator is less, then the limit is undefined.

Section 1: Activity 2

1. a. $a=1$ and $r=\frac{1}{2}$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

b. $a=3$ and $r=-2$

$$3-6+12-24+48-\ldots$$

c. $a=3$ and $r=-\frac{1}{3}$

$$3-1+\frac{1}{3}-\frac{1}{9}+\ldots$$

2. From Example 1, $a=3$ and $r=0.75$.

$$S=2+\frac{a}{1-r}$$
$$=2+\frac{3}{1-0.75}$$
$$=2+12$$
$$=14$$

From Example 2, $a=\frac{1}{2}$ and $r=\frac{1}{2}$.

$$S=\frac{a}{1-r}$$
$$=\frac{\frac{1}{2}}{1-\frac{1}{2}}$$
$$=1$$

3. a. $a=4$ and $r=\frac{1}{2}$

$$S=\frac{a}{1-r}$$
$$=\frac{4}{1-\frac{1}{2}}$$
$$=8$$

b. $a=3$ and $r=-\frac{1}{3}$

$$S=\frac{a}{1-r}$$
$$=\frac{3}{1-\left(-\frac{1}{3}\right)}$$
$$=\frac{3}{\frac{4}{3}}$$
$$=\frac{9}{4}$$

c. The infinite geometric series begins with the second term, 4.

$a = 4$ and $r = -\frac{1}{2}$

$$\begin{aligned} S &= 6 + \frac{a}{1-r} \\ &= 6 + \frac{4}{1-\left(-\frac{1}{2}\right)} \\ &= 6 + \frac{4}{\frac{3}{2}} \\ &= 6 + \frac{8}{3} \\ &= \frac{26}{3} \end{aligned}$$

d. Since $r = 2$, there is no finite sum.

$\therefore\ S = \infty$

4. a. $0.121\,212\ldots = 0.12 + .0012 + 000\,012 + \ldots$

$a = 0.12$ and $r = 0.01$

$$\begin{aligned} S &= \frac{a}{1-r} \\ &\frac{0.12}{1-0.01} \\ &= \frac{0.12}{0.99} \\ &= \frac{12}{99} \\ &= \frac{4}{33} \end{aligned}$$

b. $2.333\ldots = 2 + (0.3 + 0.03 + 0.003 + \ldots)$

$a = 0.3$ and $r = 0.1$

$$\begin{aligned} S &= 2 + \frac{a}{1-r} \\ &= 2 + \frac{0.3}{1-0.1} \\ &= 2 + \frac{1}{3} \\ &= \frac{7}{3} \end{aligned}$$

c. $6.123\,23 = 6.1 + (0.023 + 0.000\,23 + 0.000\,0023 + \ldots)$

$a = 0.023$ and $r = 0.01$

$$\begin{aligned} S &= 6.1 + \frac{a}{1-r} \\ &= 6.1 + \frac{0.023}{1-0.01} \\ &= 6.1 + \frac{0.023}{0.99} \\ &= \frac{61}{10} + \frac{23}{990} \\ &= \frac{6039}{990} + \frac{23}{990} \\ &= \frac{6062}{990} \\ &= \frac{3031}{495} \end{aligned}$$

5. Total Perimeter $= 6 + 3 + 1.5 + \ldots$

$a = 6$ and $r = 0.5$

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{6}{0.5} \\ &= 12 \end{aligned}$$

The total perimeter is 12 units.

6. Total Distance $= 30 + 30(0.9) + 30(0.9)^2 + \ldots$

$a = 30$ and $r = 0.9$

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{30}{1-0.9} \\ &= 300 \end{aligned}$$

The distance the ball travels is 300 cm.

Section 1: Follow-up Activities

Extra Help

1. a. The conjugate is $\sqrt{3} - \sqrt{2}$.

$$\begin{aligned} \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) &= 3 - 2 \\ &= 1 \end{aligned}$$

b. The conjugate is $2\sqrt{3} + \sqrt{7}$.

$$\begin{aligned} \left(2\sqrt{3}-\sqrt{7}\right)\left(2\sqrt{3}+\sqrt{7}\right) &= 4(3) - 7 \\ &= 5 \end{aligned}$$

c. The conjugate is $2\sqrt{5}-3\sqrt{7}$.

$$\begin{aligned}\left(2\sqrt{5}+3\sqrt{7}\right)\left(2\sqrt{5}-3\sqrt{7}\right)&=4(5)-9(7)\\&=-43\end{aligned}$$

Each product is rational.

2. a.
$$\begin{aligned}\frac{6}{\sqrt{7}-2}&=\frac{(6)\left(\sqrt{7}+2\right)}{\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)}\\&=\frac{6\sqrt{7}+12}{7-4}\\&=\frac{6\sqrt{7}+12}{3}\\&=2\sqrt{7}+4\end{aligned}$$

b.
$$\begin{aligned}\frac{3+\sqrt{2}}{2+\sqrt{2}}&=\frac{\left(3+\sqrt{2}\right)\left(2-\sqrt{2}\right)}{\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)}\\&=\frac{6-3\sqrt{2}+2\sqrt{2}-2}{4-2}\\&=\frac{4-\sqrt{2}}{2}\end{aligned}$$

c.
$$\begin{aligned}\frac{\sqrt{6}-2\sqrt{3}}{2\sqrt{6}+\sqrt{3}}&=\frac{\left(\sqrt{6}-2\sqrt{3}\right)\left(2\sqrt{6}-\sqrt{3}\right)}{\left(2\sqrt{6}+\sqrt{3}\right)\left(2\sqrt{6}-\sqrt{3}\right)}\\&=\frac{2(6)-\sqrt{18}-4\sqrt{18}+2(3)}{[4(6)-3]}\\&=\frac{12-5\sqrt{18}+6}{24-3}\\&=\frac{18-5\sqrt{9(2)}}{21}\\&=\frac{18-15\sqrt{2}}{21}\\&=\frac{6-5\sqrt{2}}{7}\end{aligned}$$

Enrichment

1. Answers will vary.

$$\begin{aligned}\frac{1}{2}&=\frac{0.1}{0.2}\\&=\frac{0.1}{1-0.8}\end{aligned}$$

$$\therefore\ a=\frac{1}{10}\text{ and }r=\frac{4}{5}$$

$$\therefore\ \frac{1}{2}=\frac{1}{10}+\frac{2}{25}+\frac{8}{125}+\ldots$$

$$\begin{aligned}\frac{1}{2}&=\frac{\frac{1}{3}}{\frac{2}{3}}\\&=\frac{\frac{1}{3}}{1-\frac{1}{3}}\end{aligned}$$

$$\therefore\ a=\frac{1}{3}\text{ and }r=\frac{1}{3}$$

$$\therefore\ \frac{1}{2}=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$$

$$\frac{1}{2} = \frac{\frac{1}{6}}{\frac{2}{6}}$$

$$= \frac{\frac{1}{6}}{1-\frac{2}{3}}$$

$$\therefore\ a = \frac{1}{6} \text{ and } r = \frac{2}{3}$$

$$\therefore\ \frac{1}{2} = \frac{1}{6} + \frac{1}{9} + \frac{2}{27} + \ldots$$

$$1 = \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$\therefore\ a = \frac{1}{2} \text{ and } r = \frac{1}{2}$$

$$\therefore\ 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

2. Answers will vary.

$$1 = \frac{0.9}{0.9}$$

$$= \frac{0.9}{1-0.1}$$

$$\therefore\ a = 0.9 \text{ and } r = 0.1$$

$$\therefore\ 1 = 0.9 + 0.09 + 0.009 + \ldots$$

$$1 = \frac{\frac{3}{2}}{\frac{3}{2}}$$

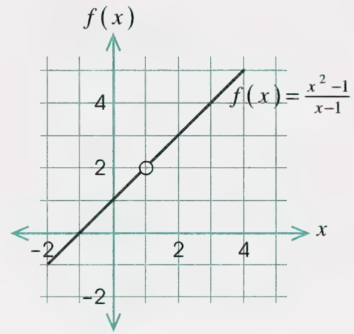
$$= \frac{\frac{3}{2}}{1-\left(-\frac{1}{2}\right)}$$

$$\therefore\ a = \frac{3}{2} \text{ and } r = -\frac{1}{2}$$

$$\therefore\ 1 = \frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \frac{3}{16} + \ldots$$

Section 2: Activity 1

1.



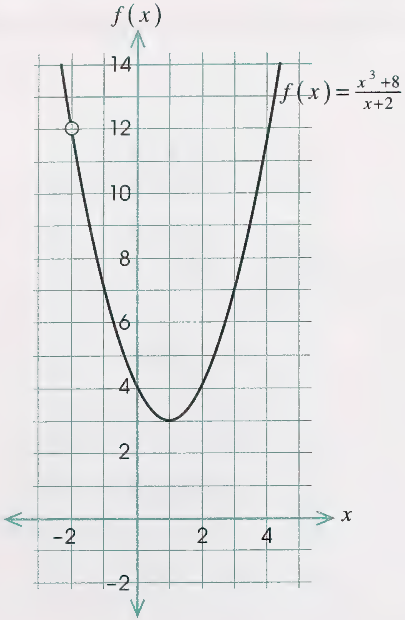
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$f(x) = x+1 \quad (\text{if } x \neq 1)$$

$$= 1+1$$

$$= 2$$

2.



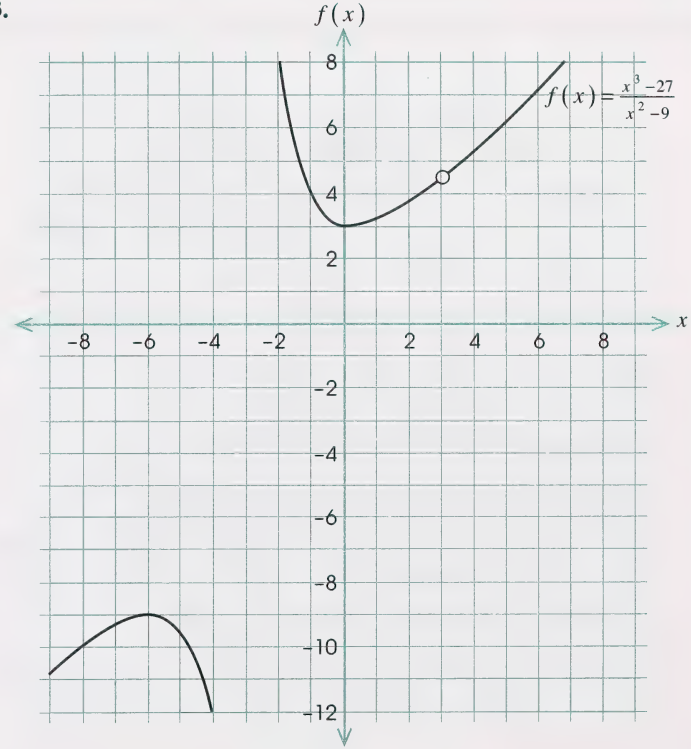
$$\lim_{x\to -2}\frac{x^3+8}{x+2}=\lim_{x\to -2}\frac{(x+2)\left(x^2-2x+4\right)}{(x+2)}$$

$$f(x)=x^2-2x+4 \quad (\text{if } x\neq -2)$$

$$=(-2)^2-2(-2)+4$$

$$=12$$

3.



$$\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\frac{(x-3)\left(x^2+3x+9\right)}{(x-3)(x+3)}$$

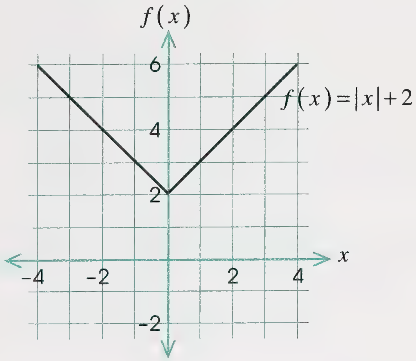
$$=\frac{3^2+3(3)+9}{3+3}$$

$$=4.5$$

Section 2: Activity 2

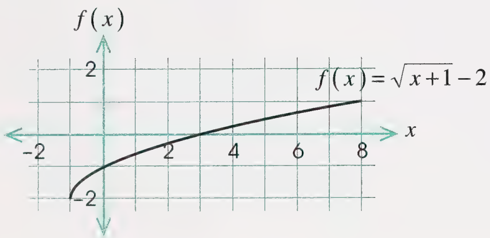
1. **a.** 0 **b.** 0 **c.** 0
   d. does not exist **e.** -2 **f.** does not exist
   g. 1 **h.** 3 **i.** does not exist
   j. 3

2. **a.**



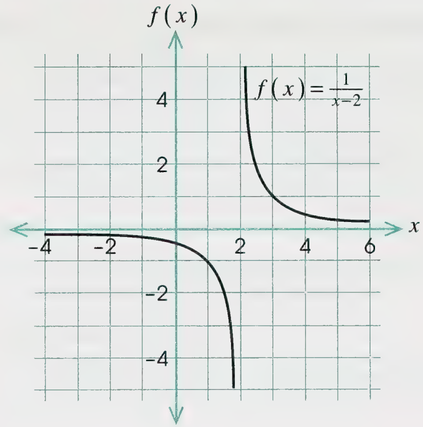
$\lim_{x\to 0} f(x)=2$

b.



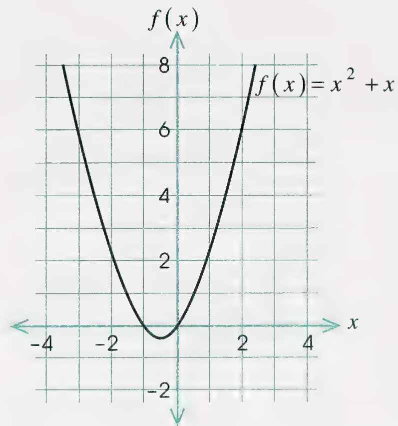
$\lim_{x\to -1^+} f(x)=-2$

c.



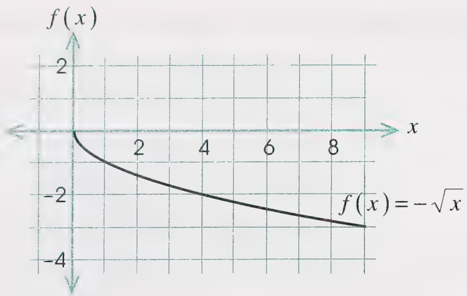
$\lim_{x\to 2^-} f(x)=-\infty$

d.



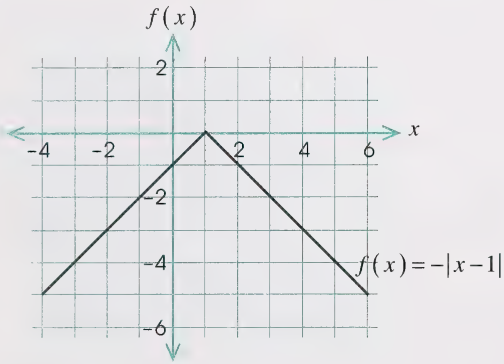
$\lim_{x\to -1} f(x)=0$

e.



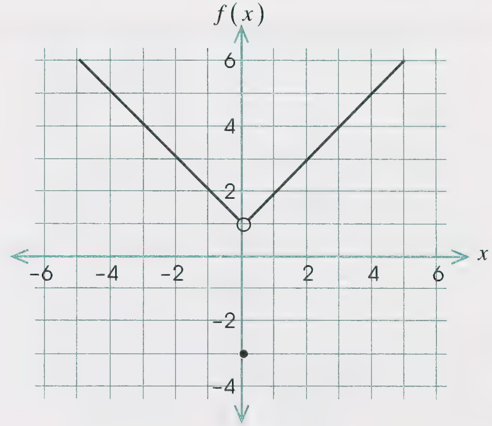
$\lim_{x \to 0^-} f(x)$ does not exist.

f.



$\lim_{x \to 1^+} f(x) = 0$

3.



a. $\lim_{x \to 0^-} f(x) = 1$

b. $\lim_{x \to 0^+} f(x) = 1$

c. $\lim_{x \to 0} f(x) = 1$

Section 2: Activity 3

1. The discontinuities in Example 1 occur at $t = 0$, $t = 1$, $t = 2$, $t = 3$, and so on since the limits of $f(x)$ do not exist at these values.

   The discontinuity in Example 2 occurs at $x = 1$ since $f(1)$ is not defined.

   The discontinuity in Example 3 occurs at $x = 2$ since the $\lim_{x \to 2} f(x)$ does not exist.

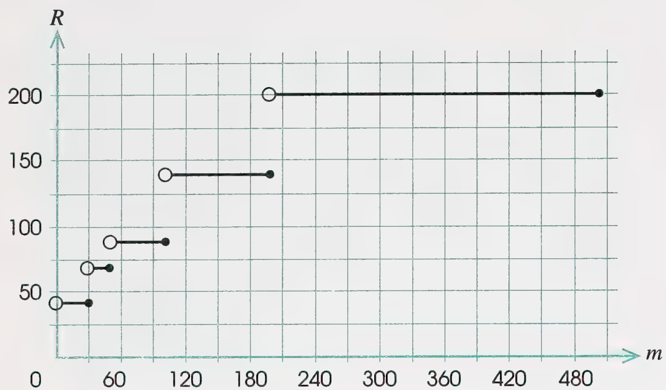
The discontinuity in Example 4 occurs at $x = 0$ since the limit of $f(x)$ does not exist at $x = 0$ and $f(0)$ is undefined.

The discontinuity in Example 5 occurs at $x = 1$ since the $\lim_{x \to 1} f(x) \neq f(1)$.

2. To find $\lim_{x \to 3} f(x)$, determine $f(3)$.

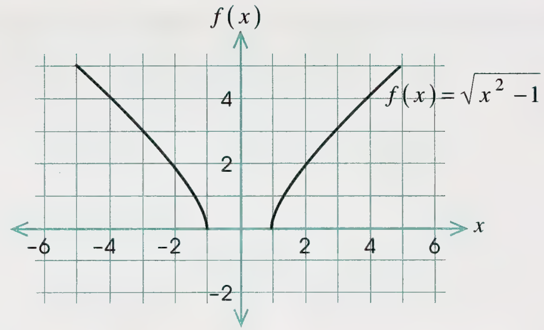
$$\begin{aligned} f(3) &= 3^2 - 1 \\ &= 8 \end{aligned}$$

3.



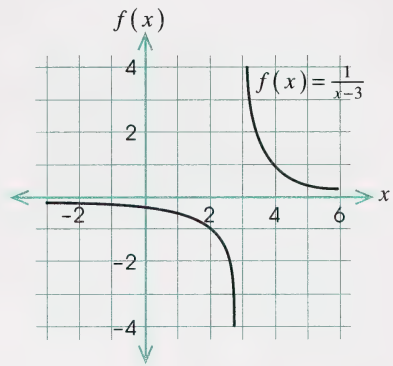
The discontinuities occur at $m = 0$ g, 30 g, 50 g, 100 g, 200 g, and 500 g. At these values, left- and right-hand limits differ.

4. a.



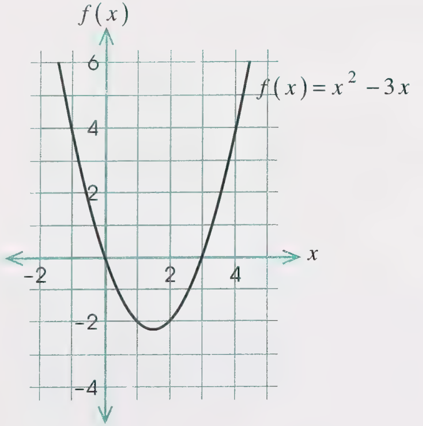
The function is discontinuous in the interval $[-1, 1]$.

b.



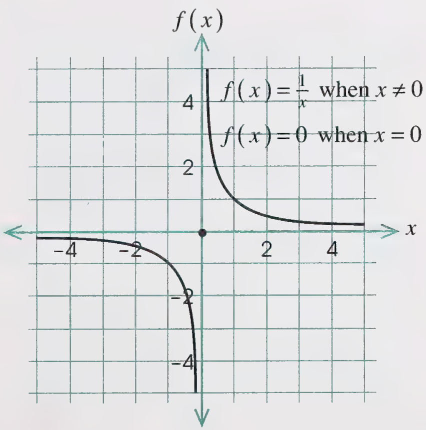
This function is discontinuous at $x = 3$.

c.



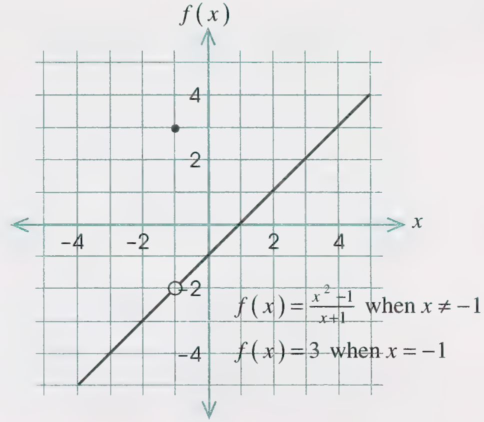
This function is continuous.

d.



This function is discontinuous at $x=0$. The left- and right-hand limits are undefined.

e.



This function is discontinuous at $x=-1$.

$$f(-1) \neq \lim_{x \to -1} f(x)$$

Section 2: Activity 4

1. a. $$\begin{aligned}\lim_{x\to 2}\left(5x^2-3\right) &= \lim_{x\to 2} 5x^2 - \lim_{x\to 2} 3\\ &= \lim_{x\to 2} 5 \bullet \lim_{x\to 2} x^2 - 3\\ &= 5\left(\lim_{x\to 2} x\right)^2 - 3\\ &= 5(2)^2 - 3\\ &= 20-3\\ &= 17\end{aligned}$$

   b. $$\begin{aligned}\lim_{x\to 0}\sqrt{5x+4} &= \sqrt{\lim_{x\to 0}(5x+4)}\\ &= \sqrt{\lim_{x\to 0} 5x + \lim_{x\to 0} 4}\\ &= \sqrt{\lim_{x\to 0} 5 \bullet \lim_{x\to 0} x + 4}\\ &= \sqrt{5\times 0+4}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

2. a.

LS	RS
$\lim_{x\to 5}\left(\frac{x^2-25}{x+5}\right)\left(\frac{x^2-25}{x-5}\right)$	$\lim_{x\to 5}\frac{x^2-25}{x+5}\bullet\lim_{x\to 5}\frac{x^2-25}{x-5}$
$=\lim_{x\to 5}\frac{\cancel{(x+5)}\cancel{(x-5)}(x+5)(x-5)}{\cancel{(x+5)}\cancel{(x-5)}}$	$=\lim_{x\to 5}\frac{\cancel{(x+5)}(x-5)}{\cancel{x+5}}\bullet\lim_{x\to 5}\frac{(x+5)\cancel{(x-5)}}{\cancel{x-5}}$
$=\lim_{x\to 5}(x+5)(x-5)$	$=\lim_{x\to 5}(x-5)\bullet\lim_{x\to 5}(x+5)$
$=10\times 0$	$=0\times 10$
$=0$	$=0$

LS = RS

b.

LS	RS
$\lim\limits_{x\to3}\dfrac{\frac{x^2-9}{x+3}}{\frac{x^2-9}{x-3}}$	$\dfrac{\lim\limits_{x\to3}\frac{x^2-9}{x+3}}{\lim\limits_{x\to3}\frac{x^2-9}{x-3}}$
$=\lim\limits_{x\to3}\dfrac{\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}}{\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}}$	$=\dfrac{\lim\limits_{x\to3}\frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}}{\lim\limits_{x\to3}\frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}}$
$=\lim\limits_{x\to3}\dfrac{x-3}{x+3}$	$=\dfrac{\lim\limits_{x\to3}(x-3)}{\lim\limits_{x\to3}(x+3)}$
$=\dfrac{0}{6}$	$=\dfrac{0}{6}$
$=0$	$=0$

LS = RS

c.

LS	RS
$\lim\limits_{x\to2}(2)\left(\dfrac{x^2-4}{x-2}\right)$	$2\lim\limits_{x\to2}\dfrac{x^2-4}{x-2}$
$=\lim\limits_{x\to2}(2)\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$	$=2\lim\limits_{x\to2}\dfrac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$
$=\lim\limits_{x\to2}2(x+2)$	$=2\lim\limits_{x\to2}(x+2)$
$=2(2+2)$	$=2(2+2)$
$=8$	$=8$

LS = RS

d.

LS	RS
$\lim\limits_{x\to1}\left(\dfrac{x^2-4x+3}{x-1}-\dfrac{x^2+x-2}{x-1}\right)$	$\lim\limits_{x\to1}\dfrac{x^2-4x+3}{x-1}-\lim\limits_{x\to1}\dfrac{x^2+x-2}{x-1}$
$=\lim\limits_{x\to1}\left[\dfrac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}}-\dfrac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}}\right]$	$=\lim\limits_{x\to1}\dfrac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}}-\lim\limits_{x\to1}\dfrac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}}$
$=\lim\limits_{x\to1}\left[(x-3)-(x+2)\right]$	$=\lim\limits_{x\to1}(x-3)-\lim\limits_{x\to1}(x+2)$
$=\lim\limits_{x\to1}(-5)$	$=(1-3)-(1+2)$
$=-5$	$=-5$

LS = RS

Section 2: Activity 5

1. Answers will vary. A correct answer must have a constant term of 0. For instance, $f(x) = x^2 - 2x$. The graph would pass through the origin.

2. Answers will vary. The denominator $h(x)$, must be nonzero for any real value of x. For instance, $f(x) = \frac{x-2}{x^2+1}$.

3. a.
$$\begin{aligned}\lim_{x\to -3} \frac{x^2-9}{x+3} &= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}} \\ &= -3-3 \\ &= -6\end{aligned}$$

b.
$$\begin{aligned}\lim_{x\to 1} \frac{x^2+7x-8}{x-1} &= \lim_{x\to 1} \frac{(x+8)\cancel{(x-1)}}{\cancel{x-1}} \\ &= 1+8 \\ &= 9\end{aligned}$$

c.
$$\begin{aligned}\lim_{x\to 3} \frac{x^2+2x-15}{x-3} &= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+5)}{\cancel{x-3}} \\ &= 3+5 \\ &= 8\end{aligned}$$

d.
$$\begin{aligned}\lim_{x\to 2} \frac{5x-10}{x-2} &= \lim_{x\to 2} \frac{5\cancel{(x-2)}}{\cancel{x-2}} \\ &= 5\end{aligned}$$

e.
$$\begin{aligned}\lim_{x\to 1} \frac{6x-2}{3x-1} &= \lim_{x\to 1} \frac{2\cancel{(3x-1)}}{\cancel{3x-1}} \\ &= 2\end{aligned}$$

f.
$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2-1} &= \lim_{x\to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2}\end{aligned}$$

g.
$$\begin{aligned}\lim_{x\to 2} \frac{x+2}{x-3} &= \frac{2+2}{2-3} \\ &= \frac{4}{-1} \\ &= -4\end{aligned}$$

h.
$$\lim_{x\to 5} \frac{x^2-x-2}{x-5} = \text{undefined}$$

Therefore, there is no limit.

i. $\lim\limits_{x\to 9} \dfrac{x^2-1}{x-9} = \text{undefined}$

Therefore, there is no limit.

j. $$\lim_{x\to 0} \frac{3-\frac{1}{x}}{\frac{2}{x}-5} = \lim_{x\to 0} \frac{\frac{3x-1}{x}}{\frac{2-5x}{x}}$$

$$= \lim_{x\to 0} \frac{3x-1}{2-5x}$$

$$= -\frac{1}{2}$$

k. $$\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}\left(x^2+3x+9\right)}{\cancel{x-3}}$$

$$= \lim_{x\to 3}\left(x^2+3x+9\right)$$

$$= 27$$

3. $$\lim_{h\to 0} \frac{(1-h)^3-1}{h} = \lim_{h\to 0} \frac{\left(1-3h+3h^2+h^3\right)-1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(-3+3h+h^2\right)}{\cancel{h}}$$

$$= \lim_{h\to 0}\left(-3+3h+h^2\right)$$

$$= -3$$

4. $$\lim_{x\to 0} \frac{5-\sqrt{25-x}}{x} = \lim_{x\to 0} \frac{\left(5-\sqrt{25-x}\right)\left(5+\sqrt{25-x}\right)}{x\left(5+\sqrt{25-x}\right)} \text{ (conjugates)}$$

$$= \lim_{x\to 0} \frac{(5)^2-\left(\sqrt{25-x}\right)^2}{x\left(5+\sqrt{25-x}\right)}$$

$$= \lim_{x\to 0} \frac{25-25+x}{x\left(5+\sqrt{25-x}\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\left(5+\sqrt{25-x}\right)}$$

$$= \lim_{x\to 0} \frac{1}{\left(5+\sqrt{25-x}\right)}$$

$$= \frac{1}{10}$$

Section 2: Activity 6

1. a. $$\lim_{x\to\infty} \frac{2}{x-2} = \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{2}{x}\right)}{\cancel{x}\left(1-\frac{2}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x}}{1-\frac{2}{x}}$$

$$= \frac{0}{1-0}$$

$$= 0$$

b. $\lim\limits_{x\to -\infty} 3^{-x} = \infty$, because as $x \to -\infty$, $-x \to \infty$

c. $\lim\limits_{x\to\infty} \dfrac{3x+2}{2x-1} = \lim\limits_{x\to\infty} \dfrac{\cancel{x}\left(3+\frac{2}{x}\right)}{\cancel{x}\left(2-\frac{1}{x}\right)}$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x}}{2-\frac{1}{x}}$$

$$= \frac{3+0}{2-0}$$

$$= \frac{3}{2}$$

d. $\lim\limits_{x\to-\infty} \dfrac{x^2-2x+1}{2x^2-3x} = \lim\limits_{x\to-\infty} \dfrac{\cancel{x^2}\left(1-\frac{2}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(2-\frac{3}{x}\right)}$

$$= \lim_{x\to-\infty} \frac{1-\frac{2}{x}+\frac{1}{x^2}}{2-\frac{3}{x}}$$

$$= \frac{1-0+0}{2-0}$$

$$= \frac{1}{2}$$

e. $\lim\limits_{x\to\infty} \dfrac{2x^2+3}{x+1} = \lim\limits_{x\to\infty} \dfrac{\cancel{x^2}\left(2+\frac{3}{x^2}\right)}{\cancel{x^2}\left(\frac{1}{x}+\frac{1}{x^2}\right)}$

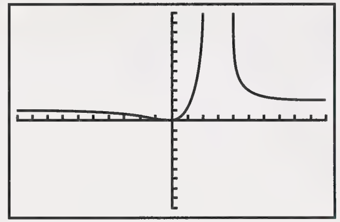
$$= \frac{2+0}{0+0}$$

$= \text{undefined}$

f. $\lim\limits_{x\to-\infty} x(x-1)^2 = -\infty$ because the factors are of the form $(-)(-)^2 = (-)$.

2. Answers will vary. One representative function is $y = \frac{3x}{1-x}$.

3. Graph X,θ,T SHIFT √ + (X,θ,T − 2) SHIFT √ EXE



$$\lim_{x\to\infty} \frac{x^2}{(x-2)^2} = \lim_{x\to\infty} \frac{x^2}{x^2-4x+4}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2}(1)}{\cancel{x^2}\left(1-\frac{4}{x}+\frac{4}{x^2}\right)}$$

$$= \frac{1}{1-0+0}$$

$$= 1$$

The asymptote is $y = 1$.

Section 2: Follow-up Activities

Extra Help

1. $x^3 - 1 = (x-1)(x^2 + x + 1)$

2. $a^3b^3 + 1000 = a^3b^3 + 10^3$
$$= (ab+10)(a^2b^2 - 10ab + 100)$$

3. $x^6 - y^3 = (x^2 - y)(x^4 + x^2y + y^2)$

4. $x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$
$$= (x+y)(x^2 - xy + y^2)(x+y)(x^2 + xy + y^2)$$

Enrichment

1. $f(x) = x + 5$

$$\begin{aligned} L &= \lim_{x \to 3}(x+5) \\ &= 3+5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} |f(x) - L| &< 10^{-2} \\ |x+5-8| &< 10^{-2} \\ |x-3| &< 10^{-2} \\ -10^{-2} < x-3 &< 10^{-2} \\ 3 - 10^{-2} < x &< 3 + 10^{-2} \\ 2.99 < x &< 3.01 \end{aligned}$$

2. $a = 0.02$ and $r = \dfrac{1}{10}$

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{(0.02)\left[1 - \left(\frac{1}{10}\right)^n\right]}{1 - \frac{1}{10}} \\ &= \frac{(0.02)\left[1 - \frac{1}{10^n}\right]}{\frac{9}{10}} \\ &= \frac{0.2}{9}\left(1 - \frac{1}{10^n}\right) \\ &= \frac{0.2}{9} - \frac{0.2}{9 \times 10^n} \end{aligned}$$

$$\begin{aligned} L &= \lim_{n \to \infty} S_n \\ &= \frac{a}{1-r} \\ &= \frac{0.02}{1 - \frac{1}{10}} \\ &= \frac{0.02}{\left(\frac{9}{10}\right)} \\ &= \frac{0.2}{9} \end{aligned}$$

$$\begin{aligned} |S_n - L| &< 10^{-8} \\ \left|\cancel{\frac{0.2}{9}} - \frac{0.2}{9 \times 10^n} - \cancel{\frac{0.2}{9}}\right| &< 10^{-8} \\ \left|\frac{-0.2}{9 \times 10^n}\right| &< \frac{1}{10^8} \\ \left|\frac{-1}{45 \times 10^n}\right| &< \frac{1}{10^8} \end{aligned}$$

Therefore, the minimum value of n is 7 (since $45 \times 10^7 > 10^8$).

NOTES

NOTES

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